Moduli of bundles on local surfaces and threefolds

Thomas Köppe

Doctor of Philosophy
University of Edinburgh
2010

Abstract

In this thesis we study the moduli of holomorphic vector bundles over a non-compact complex space X, which will mainly be of dimension 2 or 3 and which contains a distinguished rational curve $\ell \subset X$. We will consider the situation in which X is the total space of a holomorphic vector bundle on $\mathbb{CP}^1$ and ℓ is the zero section.

While the treatment of the problem in this full generality requires the study of complex analytic spaces, it soon turns out that a large part of it reduces to algebraic geometry. In particular, we prove that in certain cases holomorphic vector bundles on X are algebraic.

A key ingredient in the description of the moduli are numerical invariants that we associate to each holomorphic vector bundle. Moreover, these invariants provide a local version of the second Chern class. We obtain sharp bounds and existence results for these numbers. Furthermore, we find a new stability condition which is expressed in terms of these numbers and show that the space of stable bundles forms a smooth, quasi-projective variety.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Thomas Köppe*)

For my family

Contents

Typeset with $\LaTeX\,2_\varepsilon$ in Minion 11/19.8

Chapter 1

Preliminaries

1.1 Introduction

The aim of this thesis is to add to the understanding of the moduli of holomorphic vector bundles on non-compact complex spaces. The cases we consider are complex surfaces and threefolds which are the total spaces of bundles over $\mathbb{CP}^1$. Both these cases are interesting not only in geometry, but also in mathematical physics. Indeed, there is an extensive theory relating holomorphic vector bundles on smooth complex surfaces to instantons on the underlying real manifold, provided by Kobayashi-Hitchin correspondence. The three-dimensional case, on the other hand, is interesting in string theory, in which holomorphic bundles, or more generally coherent sheaves, describe string boundary conditions (so-called D-branes). A description of the moduli of such bundles is therefore important for any type of problem that requires integration over "all branes", which is a staple of mathematical physics.

Several results of this thesis have been published in joint work with my supervisor E. Gasparim, with physicist P. Majumdar and with E. Ballico [GKM, BGK1, BGK2, GK]. Some results will only be cited, while the proofs of others are repeated here. By and large, lots of the results on complex surfaces (Chapter 2) have been published, while the material in Chapter 3 on threefolds is new.

The Kobayashi-Hitchin correspondence between irreducible $SU(2)$-instantons and stable holomorphic vector bundles of rank 2 was first proved by Donaldson for projective surfaces, then for general compact Kähler surfaces by Uhlenbeck and Yau, for $\mathbb{C}^2$ by Donaldson again, and for $\widetilde{\mathbb{C}}^2$, the blow-up of the plane in the origin, by King. The result was extended to the non-compact spaces Z_k described below in [GKM], where $Z_1 = \widetilde{\mathbb{C}}^2$.

When passing from complex projective geometry to non-compact spaces, one immediately faces the complication that there exist holomorphic objects that are not algebraic. We will briefly review the basic definitions of the categories of complex schemes and analytic spaces, before demonstrating that the class of non-compact spaces of the form $Z_k := \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-k)\big)$ for $k > 0$ satisfies GAGA-type properties, as does the space $W_1 := \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\big)$. Armed with this knowledge, we are able to present an explicit description of holomorphic vector bundles on Z_k and W_1 and to attempt a first guess at how to parametrise their moduli.

We find that every holomorphic vector bundle of rank 2 on either Z_k or W_1 is an extension of line bundles, and thus parametrised by an Ext^1-group, which turns out to be finite-dimensional and for which we have a concrete description. The moduli of all rank-2 bundles with a fixed first Chern class (which we usually take to be zero) is therefore a quotient of a finite-dimensional vector space.

The main part of this thesis consists of the construction of several holomorphic numerical invariants of vector bundles. These techniques are applicable both in the case of surfaces and of threefolds, and indeed they generalise to spaces of higher dimensions and bundles of higher rank. The crucial condition on the base space is that it contains a contractible rational curve $\ell \cong \mathbb{P}^1$. If Z denotes any such space, we write $\pi\colon Z \to X$ for the contraction of ℓ (so for example we have $\pi\colon Z_1 = \widetilde{\mathbb{C}^2} \to \mathbb{C}^2$). If E denotes a holomorphic vector bundle on Z and $\mathcal{E} := \mathcal{O}_Z(E)$ its sheaf of holomorphic sections, then two of the numerical invariants of E are obtained from the cohomology of the direct image $R\pi_*E$. To be precise, they are the two summands of the expression

$$\chi^{\mathrm{loc}}(\ell, E) := h^0\big(X; (\pi_*\mathcal{E})^{\vee\vee}/\pi_*\mathcal{E}\big) + h^0\big(X; R^1\pi_*\mathcal{E}\big)\,. \tag{1.1}$$

We will consider the situation in which Z is the total space of a holomorphic vector bundle on $\mathbb{CP}^1$ and ℓ is the zero section. We construct further invariants (which do not necessarily require the space to contain a contractible curve, merely a compact 2-cycle) from the cohomology of the endomorphism sheaf $\mathcal{E}nd\,E$. Note that $E|_\ell$ splits as a sum of line bundles $\mathcal{O}_\ell(-j) \oplus \mathcal{O}_\ell(j)$, assuming $c_1(E) = 0$, and by considering bundles of fixed j, fixed χ^{loc} and a fixed sum decomposition as in Equation (1.1), we are able to describe the moduli of all rank-2 bundles with $c_1 = 0$.

The computation of the two numbers in χ^{loc} proceeds by iteration over infinitesimal neighbourhoods of ℓ, using the Theorem on Formal Functions (Theorem 1.6). We will discuss the distinction between the algebraic and the analytic category and conclude that we obtain the same results by performing the computations in either category. Finally, I developed a set of computer algorithms for the computation of the invariants, using the great open-source computer algebra system *Macaulay 2* by Grayson and Stillman [M2]. A detailed description of the implementation has been submitted as a separate publication. We present the somewhat technical description of the algorithms in the appendix to Chapter 2. § 2.B may be skipped without loss of continuity, and an illustrative example is worked out in § 2.A. These computer-based automated computations have been used for several results.

The main results of this thesis are:

Theorems 2.1, 3.10 and 3.11. *Let $E \to X$ be a holomorphic vector bundle, where X is one of the spaces* $\mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-k)\big)$*, $k \geq 1$, or* $\mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\big)$*. Then E is filtered, i.e. there exists a sequence $E_1 \subset \cdots \subset E_r = E$ such that E_1 is a line bundle and $E_i\big/E_{i-1}$ is a line bundle for $2 \leq i \leq r$, and moreover all bundles E_i are algebraic. If $X = \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}\big)$, then filtrability still holds, but E may be non-algebraic. In particular, if $\operatorname{rk} E = 2$, E is an extension of line bundles.*

Numerical invariants (see Sections 2.4 and 3.5). Let $E \to X$ be a holomorphic vector bundle of rank 2 with $c_1(E) = 0$. If $X = \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-k)\big)$, $k \geq 1$, and $\mathbb{P}^1 \cong \ell \subset X$ denotes the contractible curve, then there exist holomorphic invariants $w(E)$ and $h(E)$ called the "width" and "height" of E, respectively, such that $w(E) + h(E)$ is the local holomorphic Euler characteristic of E near ℓ in the sense of Blache [Bl96]. We prove sharp bounds for $w(E)$ and $h(E)$, see Results 2.14, 2.15 and 2.17. As a consequence, we find that for $k > 2$, the local holomorphic Euler characteristic (and by correspondence, the instanton charge) cannot assume every positive integer and instead has gaps. In [GKM] we interpreted this as the fact that the self-intersection number of ℓ provides an obstruction to instanton decay.

If X is the threefold $\mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\big)$, we can also define $w(E)$ and $h(E)$, but we show that $w(E)$ always vanishes. However, we construct new invariants from the cohomology of $\mathcal{E}nd\, E$. For $X = \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}\big)$ we refine our approach to obtain numerical invariants and deal with inevitable infinities that arise from the non-compactness of X.

Theorems 2.26 and 3.31 (Sketch statement). *Let $E \to X$ be a holomorphic vector bundle of rank 2 with $c_1(E) = 0$ and X be as above, and $E|_\ell \cong \mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$, $j \geq 0$. Write $\mathfrak{M}(X; j)$ for the collection of all such bundles modulo isomorphism, which has a natural topology of a quotient of a vector space. When X is a surface, there is an embedding $\Phi\colon \mathfrak{M}(j+1) \hookrightarrow \mathfrak{M}(j+1)$. When X is the threefold, there is a family of embeddings $\Phi_{[a:b]}\colon \mathfrak{M}(j+1) \hookrightarrow \mathfrak{M}(j+1)$ parametrised by a linear system $\cong \mathbb{P}^1 \ni [a:b]$.*

We will use these last results to describe the structure of the full moduli of rank-2 bundles with vanishing first Chern class by induction on j.

Acknowledgements. I am greatly indebted to Elizabeth Gasparim for confronting me with fascinating and challenging questions and for guidance in matters mathematical and meta-mathematical. I am also most grateful to my collaborators Pushan Majumdar and Edoardo Ballico for an exciting entry into mathematical research, to Mike Stillman and Dan Grayson for creating and maintaining an active community around *Macaulay 2*, and to Irena Swanson for interesting discussion, inspiration with the computer algorithms and editorial work. Fruitful discussions were also had with Alexey Bondal, Alastair Craw, Gavin Brown, Tom Bridgeland, Jean-Paul Brasselet, Sheldon Katz, Jonathan Block, Patrick Clarke and Tony Pantev, and with Shiying Dong and Artan Sheshami, while Toby Bailey provided vital coffee and Andrew Ranicki essential IT resources.

I should also like to thank the Engineering and Physical Sciences Research Council and the London Mathematical Society for their support and the Centre International de Rencontres Mathématiques, the University of Pennsylvania and the Indian Association for the Cultivation of Science for their hospitality, all of which contributed significantly to this work.

1.2 Analytic and algebraic geometry

The objects of our study lie at the confluence of different fields of mathematics, namely topology, differential geometry, analysis and algebra. To study the geometry of a space X, we will need to know its topology and its differential structure, so the notion of smooth manifolds and vector bundles enters, but this is not quite enough. To fully express the subtleties that arise, we need the notion of coherent sheaves over schemes and analytic spaces, or even over formal schemes and formal spaces (and morally, but nothing else, stacks).

To begin, we will introduce two related notions of *analytic spaces* and *schemes*. To this end, we first define several basic algebraic notions.

1.2.1 Basic definitions

We assume familiarity with basic notions of group and ring theory, which may be found in standard textbooks such as [Lan02]. As usual, a *ring* is a set R with a commutative, associative binary operation $+$ with identity and inverses and another associative binary operation $\times$ which distributes over $+$. We call R *commutative* if $\times$ is commutative, and we say that R *has a unit* if $\times$ has a unit. Commutative rings with unit will be the most important types of rings for us.

Assuming that R is commutative, an *R-module* is a set M with a commutative, associative binary operation $+$ with identity and inverses and an operation $R \times M \to M$ (called *scaling*) which preserves + (that is, $r.(x+y) = r.x + r.y$ and $(r+s).x = r.x + s.x$ for $r, s \in R$, $x, y \in M$) and for which $r.(s.x) = (r \times s).x$ and $1_R.x = x$ if R has a unit. If R were not commutative, we would have to distinguish left and right actions of R on M, but we will not need this.

Commutative rings with unit form a *category*, that is, a collection of *objects* (namely the rings) together with a collection of *morphisms*, namely the structure-preserving maps between rings, or *ring homomorphisms*. There is a special ring which has a unique map into every other ring, namely the integers $\mathbb{Z}$, and the unique map $\mathbb{Z} \to R$ sends 1 to 1_R. Such an object is called *initial*, and this fact is important for the theory of schemes.

Related to modules are algebras: An *algebra* over R is an R-module M which has itself another binary operation $*$ (not necessarily associative or commutative, but conventions differ) which is bilinear with respect to the scaling action of R: $(r.x + s.y) * z = r.(x * z) + s.(y * z)$ and $x * (r.y + s.z) = r.(x * y) + s.(x * z)$.

In fact, every Abelian group is a $\mathbb{Z}$-module and every ring with unit is a $\mathbb{Z}$-algebra, so if one sought to minimise the number of definitions, modules, algebras and $\mathbb{Z}$ alone would encompass all essential notions of basic group and ring theory. Recall that a *field* is a commutative ring with unit for which $\times$ has inverses. The most important field for us will be the field of complex numbers $\mathbb{C}$.

The study of modules and (commutative) algebras over commutative rings is known as "Commutative Algebra", and its importance (for us) lies in the fact that commutative algebras capture the local nature of geometric spaces. To see this, note that if X is any topological space, then the set of continuous $\mathbb{C}$-valued functions on X, $C(X) := \{f \colon X \to \mathbb{C}\}$, forms a commutative ring, the ring structure given pointwise, and points of X correspond to *maximal ideals*

$$I_x := \left\{ f \in C(X) : f(x) = 0 \right\} .$$

Thus a commutative ring encodes information about a topological space. (In fact, a result by Gelfand and Naimark says that one can recover X from $C(X)$ when X is compact.) While the ring $C(X)$ is rather large and unwieldy, corresponding constructions for rings of analytic, holomorphic or algebraic functions on a suitable space yield very well-understood commutative rings and are one of the cornerstones of geometry.

The remainder of this section serves to fix notation and conventions and set the mood rather than give complete and exhaustive definitions. The material is standard, good textbook references are [Ei95] for commutative algebra, [Har77] for algebraic geometry and sheaves, and Cartan-Eilenberg or Gelfand-Manin for homological algebra and category theory.

Algebra. We will write $\mathbb{C}\{x_1,\dots,x_n\}$ for the $\mathbb{C}$-algebra of power series in n variables that converge on a neighbourhood of $0 \in \mathbb{C}^n$. If $\Bbbk$ is any field (or indeed commutative ring with unit), we will write $\Bbbk[x_1,\dots,x_n]$ for the $\Bbbk$-algebra of polynomials in n variables. Clearly $\mathbb{C}[x_1,\dots,x_n] \subsetneq \mathbb{C}\{x_1,\dots,x_n\}$.

We will also reserve the notation $R, S, \dots$ for commutative rings with unit. It is convenient to think of the collection of modules over a fixed base ring R in categorical terms. We denote by $\mathfrak{Mod}_R$ the (Abelian) category of R-modules and by $\mathfrak{mod}_R$ the full subcategory of finitely generated R-modules. The property of being Abelian is one which requires rather too much category theory for us to warrant making precise, but the category of modules over a commutative ring, or in particular the category of Abelian groups, are the prototypical examples of an Abelian category. The important features are the existence of a zero object, direct sums, kernels, cokernels, and a canonical isomorphism $\ker \operatorname{coker} f \cong \operatorname{coker} \ker f$ for every morphism $f\colon A \to B$; this allows us to speak of *exact sequences* in the category: $A \xrightarrow{f} B \xrightarrow{g} C$ is *exact at B* if $\ker g = \operatorname{coker} f$.

If $\mathcal{A}$ is an Abelian category, we write $C(\mathcal{A})$ for the (Abelian) category of cochain complexes, $K(\mathcal{A})$ for the (Abelian) category of cochain complexes up to cochain homotopy, and $D(\mathcal{A})$ for the derived category of $\mathcal{A}$. We also write $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ for the full subcategories of bounded (respectively above, below and both) complexes, and for $C(\mathcal{A})$ and $D(\mathcal{A})$ similarly. Note that $K(\mathcal{A})$ and $D(\mathcal{A})$ are naturally triangulated. We write $C_{\mathfrak{mod}_{\mathfrak{R}}}(\mathfrak{Mod}_R)$ for the category of cochain complexes in $\mathfrak{Mod}_R$ whose cohomologies lie in $\mathfrak{mod}_R$, and similarly for K and D.

The key idea here is the notion of *exactness* in an Abelian category. We would like functors to preserve the structure of a category. Most real-world functors on our standard categories (say $\mathfrak{Mod}_R$) such as tensor product and Hom preserve direct sums, i.e. the *additive* structure of the category, but not exactness, i.e. the *Abelian* structure. That is, we have additive functors on additive categories, but rarely "Abelian" functors on Abelian categories. To remedy the situation, one introduces the derived category of an Abelian category. The derived category is no longer Abelian, but *triangulated*, and its natural choice of structure-preserving functors are those which preserve distinguished triangles – let us call those *triangular* functors. The derived category has the motivating property that (subject to certain conditions) an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between Abelian categories $\mathcal{A}$ and $\mathcal{B}$ induces a triangular functor $RF\colon D^b(\mathcal{A}) \to D^b(\mathcal{B})$, and restricted to $\mathcal{A}$, RF maps exact sequences to distinguished triangles.

The functor RF is called the *right-derived functor* of F, and it requires that F be left-exact. There is a dual notion of a *left-derived functor LF* when F is *right*-exact. We concentrate on the right-derived version only, since the dual notion is no more complicated and the right-derived version is far more prevalent in our applications. (Most notably, the direct image sheaf functor

f_* is left-exact, and its right-derived functor Rf_* will play a central role, for instance in the guise of sheaf cohomology.)

Sheaves. We must retain the language of categories for a little longer in order to introduce *sheaves*, and all-important tool in geometry. The concept of a sheaf generalises several notions of topology and geometry, and while the definition is somewhat high-brow, two concrete examples illustrate the motivation for sheaves:

- *Constructible* sheaves and in particular *locally constant* sheaves in algebraic topology: For example, singular cohomology $H^*(X;\mathbb{Z})$ with coefficients in $\mathbb{Z}$ is not good at dealing with non-orientable spaces, but replacing $\mathbb{Z}$ with the locally constant orientation sheaf (sometimes denoted $\{\mathbb{Z}\}$) absorbs the problem of orientability. As another example, singular (co)homology has Poincaré duality on orientable manifolds, but by replacing $\mathbb{Z}$ with a suitable constructible sheaf, we can extend Poincaré duality to spaces that are merely *stratified* by manifolds (Poincaré-Verdier duality).

  We will not need constructible sheaves in this thesis.

- *Coherent* sheaves generalise the notion of vector bundles. If $f\colon E \to F$ is a map of vector bundles, there is in general no cokernel of f which is also a bundle. However, every vector bundle corresponds to a coherent sheaf, and the category of coherent sheaves over a fixed space does have cokernels – in fact, it is an Abelian category.

We will now turn this motivation into a precise definition.

If X is any topological space, there is a category $\mathfrak{Open}_X$ whose objects are the open sets of X and whose morphisms are the inclusions. A *(set-valued) presheaf* on X is a functor $\mathcal{F} \in [\mathfrak{Open}_X^{\mathrm{op}}, \mathfrak{Set}]$, where $\mathfrak{Set}$ is the category of sets, $\mathcal{C}^{\mathrm{op}}$ denotes the opposite category of a category $\mathcal{C}$ and $[\mathcal{C},\mathcal{D}]$ denotes the category of functors from a category $\mathcal{C}$ to a category $\mathcal{D}$, whose morphisms are natural transformations. A *presheaf of Abelian groups* takes values in the concrete category of Abelian groups, and a *presheaf of rings, modules, algebras* etc. takes values in the respective concrete subcategories. A presheaf is a *sheaf* if the *gluing axiom* holds: For any two open subsets $U, V \subseteq X$, if there exist $s_1 \in \mathcal{F}(U)$ and $s_2 \in \mathcal{F}(V)$ such that if $s_1|_{U\cap V} = s_2|_{V\cap U}$, then there exists $t \in \mathcal{F}(U \cup V)$ such that $t|_U = s_1$ and $t|_V = s_2$. For every point $x \in X$, the *stalk at x* of a presheaf $\mathcal{F}$ is

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U) \,,$$

i.e. elements of $\mathcal{F}_x$ are represented by pairs (V, s_V) where V is open and contains x, $s_V \in \mathcal{F}(V)$, and $(V', s_{V'})$ is equivalent to (V, s_V) if and only if $x \in V \cap V'$ and $s_V|_{V \cap V'} = s_{V'}|_{V' \cap V}$.

Some natural ways of associating a set or an Abelian group to each open set $U \subseteq X$ do not define a sheaf, but only a presheaf. There is a canonical way to associate a sheaf $\mathcal{F}^+$ to each presheaf $\mathcal{F}$, called *sheafification*, such that if $\mathcal{F}$ is already a sheaf, then $\mathcal{F}^+ = \mathcal{F}$, and such that $\mathcal{F}^+_x = \mathcal{F}_x$ for all $x \in X$. One way of doing so is to realise that for a map of topological spaces $f\colon Y \to X$, the assignment $U \mapsto \{s\colon U \to Y : s \circ f = \mathrm{id}_U\}$ defines a sheaf, the *sheaf of sections of* f, and that *every* sheaf is of this form. That is, for every sheaf $\mathcal{F}$ on X there exists a map $E_{\mathcal{F}} \to X$ whose sheaf of sections is $\mathcal{F}$. The space $E_{\mathcal{F}}$ is called the *étale space* of $\mathcal{F}$. However, the étale space exists even for all presheaves (and is constructed by gluing together all the stalks), and the sheaf of sections of $E_{\mathcal{F}} \to X$ is the sheafification of $\mathcal{F}$.

If $\mathcal{F}$ is a sheaf of Abelian groups on X, then $\Gamma(\mathcal{F}) := \mathcal{F}(X)$ denotes the group of *global sections*, and Γ is an additive functor from the category of Abelian sheaves on X to the category of Abelian groups. Its right-derived functors $R^i\Gamma$ are the *sheaf cohomology groups*,

$$H^i(X; \mathcal{F}) := R^i\Gamma(\mathcal{F}) .$$

They measure the extent to which taking global sections is not exact.

1.2.2 Geometric spaces

Having outlined the necessary machinery of sheaves, we can now make a precise definition of what we mean by a "space". This includes the basic notion of manifolds, which are topological spaces locally homeomorphic to $\mathbb{R}^n$, but also extends to spaces with singularities and spaces with "multiple points", like the zero locus of $x \mapsto x^2$. As we said earlier, the guiding idea is to describe a space by describing a suitable collection of functions on the space. But since there are often not enough functions defined on a space, we replace global functions by a sheaf of local functions.

Definition 1.1. A *ringed space* is a pair $(X, \mathcal{A})$, where X topological space and $\mathcal{A}$ is a sheaf of commutative rings with unit on X. A *locally ringed space* is a ringed space $(X, \mathcal{A})$ where each stalk $\mathcal{A}_x$ is a *local ring*, i.e. a ring with a unique maximal ideal, which we denote by $\mathfrak{m}_x$. We will also write $\mathcal{A} =: \mathcal{O}_X$ and call $\mathcal{O}_X$ the *structure sheaf* of X.

Definition 1.2. If $(X, \mathcal{A})$ is a ringed space, we say that a sheaf $\mathcal{F}$ on X is an $\mathcal{A}$*-module* if each $\mathcal{F}(U)$ is an $\mathcal{A}(U)$-module and the induced maps are module homomorphisms. We call an

$\mathcal{A}$-module $\mathcal{F}$ *locally free* if $\mathcal{F}(U)$ is a free $\mathcal{A}(U)$-module for all open sets $U \subseteq X$; equivalently if all stalks $\mathcal{F}_x$ are free $\mathcal{A}_x$-modules.

This definition already encompasses topological and smooth manifolds, where the structure sheaf is the sheaf of germs of continuous or smooth functions, respectively, and points corresponds to maximal ideals of germs vanishing at that point. So far this gives us nothing new, though, since a topological space determines its sheaf of germs of continuous functions. That is to say, if $(X, \mathcal{O}_X)$ is a topological space and $U \subseteq X$ is an open subset, then every section $s \in \mathcal{O}_X(U)$ corresponds precisely to a function $[s]\colon U \to \mathbb{R}$, where $[s](x)$ denotes the image of s in $\mathcal{O}_{X,x}/\mathfrak{m}_x\mathcal{O}_{X,x}$.

The power of the theory of locally ringed spaces comes from allowing more general sheaves than just rings of functions, so that a section s of the structure sheaf contains more information that just its values $[s](x)$.

Example. Consider the ringed space $(X, \mathcal{O}_X)$ where $X = \{*\} = \{z = 0\} \subset \mathbb{C}$ is a single point and $\mathcal{O}_X = \mathbb{C}[z]/(z^n)$. The underlying topological space contains just one point, but a section of the structure sheaf is

$$s = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} \in \mathcal{O}_X(X) \cong \mathbb{C}^n \ .$$

That is, a "function" on X in the ringed-space sense is given by n numbers, namely the first $n-1$ derivatives at $z = 0$ of a function on $\mathbb{C}$. The topological space only sees the continuous function $[s](0) = a_0$. This space is an n-fold point. It has distinct ringed subspaces of k-fold points for all $k < n$.

Analytic spaces. Note that the algebra $\mathbb{C}\{x_1, \dots, x_n\}$ is a local algebra, i.e. it has a unique maximal ideal (namely the ideal of power series without constant term). We call an algebra $\mathcal{A}$ a $\mathbb{C}$-*analytic algebra* if it is isomorphic to the quotient of $\mathbb{C}\{x_1, \dots, x_n\}$ by some finitely generated ideal for some n. (A similar notion exists for real-analytic algebras.)

Definition 1.3. A $\mathbb{C}$-*analytic space* is a locally ringed space $\left(X, \mathcal{O}_X\right)$ where each stalk $\mathcal{O}_{X,x}$ is a $\mathbb{C}$-analytic algebra. The structure sheaf $\mathcal{O}_X$ is the sheaf of germs of holomorphic functions, whose stalks consist of power series that converge on some neighbourhood.

Remark 1.4. Every analytic space looks locally like $\left\{f_1 = \cdots = f_r = 0\right\} \subseteq U$, where $U \subseteq \mathbb{C}^n$ is a polydisk neighbourhood of the origin and the f_i are holomorphic functions on U, and the corresponding analytic algebra is just $\mathbb{C}\{x_1, \dots, x_n\}/(f_1, \dots, f_r)$. In fact, the *anti-equivalence* *principle* says precisely that *germs of analytic spaces* correspond precisely to analytic algebras. We call the induced topology on X the *Euclidean* or *analytic* topology.

Schemes. (See standard textbooks like [Har77].) To every commutative ring with unit R we can associate a locally ringed space $\bigl(\operatorname{Spec}(R) = X, \mathcal{A}\bigr)$ such that $\mathcal{A}(X) = R$; this space $\operatorname{Spec}(R)$ is called an *affine scheme*. Its points are the prime ideals of R, and its *closed* points are the maximal ideals. The topology coming from the Spec-construction is the *Zariski topology*, in which closed sets are precisely the zero locus of polynomials.

A general *scheme* is covered by open sets that are affine schemes. If R is the quotient of $\mathbb{C}[x_1, \dots, x_n]$ by an ideal (all such ideals are finitely generated by Hilbert's basis theorem), we say that $\operatorname{Spec}(R)$ is an affine scheme over $\mathbb{C}$ (and similarly for general schemes). Note that such a scheme over $\mathbb{C}$ is a locally ringed space whose structure sheaf is the *sheaf of germs of regular functions*, whose stalks are germs of *polynomial functions*. Equivalently, such a scheme is locally the zero locus $\bigl\{f_1 = \cdots = f_r = 0\bigr\} \subset \mathbb{C}^n$ of *polynomials*.

Analytification. Since a polynomial ring over $\mathbb{R}$ or $\mathbb{C}$ is contained in the ring of convergent power series and the latter is a module over the former, every real or complex scheme defines uniquely a real- or complex-analytic space, which we may call the *analytification* of the scheme.

Remark 1.5. The open unit disk $\Delta \subset \mathbb{C}$ is a complex-analytic space that is not the analytification of any complex scheme of finite type.

Formal spaces and schemes. If $\bigl(X, \mathcal{O}_X\bigr)$ is a complex space or scheme and $\mathcal{I} \subset \mathcal{O}_X$ a sheaf of ideals defining a subspace $A \subseteq X$, then the locally ringed space

$$A^{(m)} := \bigl(A, \mathcal{O}_X / \mathcal{I}^{m+1}|_A\bigr)$$

is called the m^{th} *infinitesimal neighbourhood* of A in X; it is itself respectively a complex space or scheme. Moreover, for varying m these neighbourhoods form an inverse system $\cdots \to A^{(m)} \to A^{(m-1)} \to \cdots \to A^{(0)} = A$. We call the inverse limit of this system the *formal completion of X along A*, written $\widehat{A}$. Note that when $\mathcal{I} = 0$, then $A = X$ and $\widehat{X} = X$. We will colloquially call $\widehat{A}$ the *formal neighbourhood of A*.

For example, the formal completion of the origin in affine n-space is given by the limit

$$\varprojlim_m \mathbb{C}[x_1, \dots, x_n] \big/ (x_1, \dots, x_n)^{m+1} = \mathbb{C}[\![x_1, \dots, x_n]\!]\,,$$

the ring of *formal* power series in n variables. By analogy with the Spec-construction $\mathbb{C}^n$ = Spec $\mathbb{C}[x_1, \dots, x_n]$, we also speak of a *formal spectrum* and write $\widehat{0} = \operatorname{Spf} \mathbb{C}[\![x_1, \dots, x_n]\!]$. Finally, a *formal complex space* or a *formal scheme* is a space that is covered by open sets that are formal spectra. In other words, formal spaces or schemes look locally like the formal completion of a space along a subspace. By virtue of our earlier remark, every complex space is also a formal complex space, and likewise for schemes.

Note as an aside that the notion of formal completion is always available in Algebraic Geometry, over any ground field, while the notion of analyticity and convergence exist mainly over $\mathbb{R}$ or $\mathbb{C}$ (and other topological fields like $\mathbb{Q}_\ell$ which come up in arithmetic algebraic geometry).

We conclude with the important Theorem on Formal Functions, also called Grauert's Comparison Theorem, which we will require for our computations (see [GPR94, p. 164]):

Theorem 1.6. *Let $f\colon X \to Y$ be a proper map of complex spaces and $\mathcal{F}$ a coherent sheaf on X. For $y \in Y$ let $\ell := f^{-1}(y)$. Then*

$$\left(R^i f_* \mathcal{F}\right)^\wedge_y \cong \varprojlim_n H^i\left(\ell^{(n)}; \mathcal{F}|_{\ell^{(n)}}\right).$$

1.3 Moduli and deformation theory

1.3.1 A short introduction

The *moduli problem* is, in a very general sense, the question whether there exists an object $\mathfrak{M}$, the *moduli*, that parametrises all objects of a certain type up to isomorphism – for example, all vector bundles over a scheme X, or all collections of subschemes of X. In that case, each point of $\mathfrak{M}$ is one such object, e.g. the isomorphism class of a vector bundle, or the subscheme. But there is more: We say that $\mathfrak{M}$ is a *fine moduli* if there is a *universal* object over $\mathfrak{M} \times X$ whose restriction to $\{m\} \times X$ is precisely the object parametrised by $m \in \mathfrak{M}$. For example, a hypothetical fine moduli of bundles on X would have a universal bundle over $\mathfrak{M} \times X$, and a fine moduli of closed subschemes of X would have a universal subscheme of $\mathfrak{M} \times X$ (the latter actually exists and is called the "Hilbert scheme of X", at least when X is sufficiently well-behaved scheme).

This situation is particularly interesting if $\mathfrak{M}$ itself has geometric structure: In good cases, when parametrising algebraic objects over a scheme, $\mathfrak{M}$ might be a scheme itself, or a more general object like an *algebraic space* or a *stack*, notions which were invented precisely to describe the solutions of moduli problems.

The language of category theory allows us to express the moduli problem more precisely: Let us define a *problem* to be a contravariant functor $F\colon \mathfrak{Schm}_A \to \mathfrak{Set}$ from a category of schemes (over some fixed base scheme A, say) to the category of sets. For each scheme X, the set $F(X)$ is the set of interest – for example, $F(X)$ could be the set of subschemes of X, or the set of vector bundles on X. We say that a scheme M is a *fine moduli* for the problem F if M represents F, that is if the functor of points $\mathrm{Hom}_{\mathfrak{Schm}_A}(-, M)$, which maps X to the set of morphisms $X \to M$, is naturally isomorphic to F. Applying the natural isomorphism to $\mathrm{id}_M \in \mathrm{Hom}_{\mathfrak{Schm}_A}(M, M)$ yields an element u in $F(M)$, which is precisely the universal object, and thus every map $f\colon X \to M$ corresponds precisely to the object $f^*u \in F(X)$.

For the weaker notion of a *coarse moduli* for the problem F we forgo representability and merely demand a natural transformation $\phi\colon F \to \mathrm{Hom}_{\mathfrak{Schm}_A}(-, M)$ such that $\phi\colon F(\operatorname{Spec} A) \to \mathrm{Hom}_{\mathfrak{Schm}_A}(\operatorname{Spec} A, M)$ is a bijection, i.e. that the elements of $F(\{\text{pt.}\})$ biject with the A-valued points of M, and such that every natural transformation $F \to \mathrm{Hom}_{\mathfrak{Schm}_A}(-, N)$ factors uniquely through ϕ. For all practical purposes, we may think of a coarse moduli for a problem F and a fixed scheme X is an abstract set bijecting with $F(X)$.

An immediate consequence of the existence of a fine moduli scheme is that the objects in $F(X)$ cannot have non-trivial automorphisms. Since real-world objects often do have automorphisms, many interesting problems do not have fine moduli. In some cases the situation can be remedied by leaving the category of schemes and considering more general categories such as algebraic spaces and stacks. However, for our purposes we will not require universal families, and we will think of a moduli scheme (or space or set) parametrising isomorphism classes of objects.

Suppose now that $\mathfrak{M}$ parametrises vector bundles over a fixed base space X up to isomorphisms, so that we may write $[E] \in \mathfrak{M}$ for the point that parametrises all bundles isomorphic to $E \to X$. If $\mathfrak{M}$ is smooth at $[E]$, the tangent space $T_{[E]}\mathfrak{M}$ measures infinitesimal first-order deformations of E. Intersection theory tells us what $T_{[E]}\mathfrak{M}$ is (if $\mathfrak{M}$ has a perfect obstruction theory[1]), and in the case of vector bundles over a projective scheme X it will be $H^1(X; \mathcal{E}nd\, E)$ (see [HL97, Cor. 4.5.2]. The upshot is that the dimension of this cohomology group is the dimension of the component of the moduli containing $[E]$.

[1]We do not want to delve too far into the fascinating theory of deformations and obstructions at this point and refer the interested reader to [BF97]. Briefly, if X is a suitably nice space (e.g. a Deligne-Mumford stack locally of finite type over $\mathbb{C}$), then the *cotangent complex* $L_X^\bullet$ is canonically defined in the derived category, and a *perfect obstruction theory* on X is map $\mathcal{E}^\bullet \to L_X^\bullet$ in the derived category satisfying certain properties. This data defines a cycle $[X]^{\mathrm{vir}}$, the so-called *virtual fundamental class* of X with respect to $\mathcal{E}^\bullet$, whose dimension is $\operatorname{rk} H^0(\mathcal{E}^\bullet) - \operatorname{rk} H^{-1}(\mathcal{E}^\bullet)$, the so-called *virtual dimension* of X (with respect to $\mathcal{E}^\bullet$), and $H^0(\mathcal{E}^\bullet) - H^{-1}(\mathcal{E}^\bullet)$ is the *virtual tangent bundle* of X.

Let us be more specific. By a *deformation* of some object Y we mean another, larger object $\mathcal{Y}$ along with a morphism $\pi\colon \mathcal{Y} \to S$ to some parametrising pointed object $(S, 0 \in S)$, such that $\mathcal{Y}_0 := \pi^{-1}(0) \cong Y$. We call $\mathcal{Y}_0$ the *central fibre* of the family π. When $S = \operatorname{Spec} \mathbb{C}[x]/(x^2)$ is the double point, we call π a *first-order deformation*. Similarly, we have higher-order deformations over $\operatorname{Spec} \mathbb{C}[x]/(x^n)$ and formal deformations over $\operatorname{Spec} \mathbb{C}[\![x]\!]$ – but note that a formal deformation does not imply that an actual deformation exists, which is essentially asking for a formal power series to converge.

For example, in the category of schemes or of analytic spaces, a very popular deformation is a *flat smoothing*, which means that π is a flat morphism (which is a homological condition) and that the non-central fibres $\mathcal{X}_s$, $s \neq 0$ are smooth. If X is not smooth and a flat smoothing exists, then one can replace the study of the complicated object $\mathcal{X}_0$ by that of a smooth object $\mathcal{X}_s$, as long as one is concerned with properties that are invariant under flat deformations (like the Hilbert polynomial).

1.4 Some results from deformation theory

Remark 1.7. If $X \subset W$ is a subspace of a complex manifold W such that the conormal sheaf $N^*_{X,W}$ is ample, the deformation space of a bundle on $\widehat{X}$ is finite-dimensional:

Fix an integer $m \geq 0$, a vector bundle E_m on $X^{(m)}$ and set $E_0 := E_m|_X$. If

$$h^2\big(X; \mathcal{E}nd\, E_0 \otimes S^m(N^*_{X,W})\big) = 0\,,$$

then there exists a vector bundle E_{m+1} on $X^{(m+1)}$ such that $E_{m+1}|_{X^{(m)}} \cong E_m$ ([Pet81, Satz 1]).

Now let F be a vector bundle over X such that $h^2\big(X; \mathcal{E}nd\, F \otimes S^t(N^*_{X,W})\big) = 0$ for all $t > 0$. If $N^*_{X,W}$ is ample, then $h^1\big(X; \mathcal{E}nd\, F \otimes S^t(N^*_{X,W})\big) = 0$ for $t \gg 0$, and hence

$$\gamma = \sum_{t\geq 0} \gamma_t = \sum_{t \geq 0} h^1\big(X; \mathcal{E}nd\, F \otimes S^t(N^*_{X,W})\big) < +\infty\,. \tag{1.2}$$

Then there exists a vector bundle G on $\widehat{X}$ such that $G|_X \cong F$, and for a fixed such G the *deformation space* of G is isomorphic to $\mathbb{C}^\gamma$ ([Pet82, Satz 2], and first Bemerkung at p. 115, and see also [dJP00, Theorem 10.3.16]). There is a vector bundle A on an analytic neighbourhood U of X in W such that $A|_{\widehat{X}} = G$, and hence $A|_X \cong F$ ([Pet82, Satz 3]).

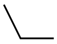
Chapter 2

Surfaces

2.1 Introduction

In this chapter we focus on complex surfaces that contain an embedded curve with negative self-intersection number. We have in mind the situation where the curve $\ell \cong \mathbb{P}^1$ inside the surface Z has the normal sheaf $N_{\ell/Z} \cong \mathcal{O}_{\mathbb{P}^1}(-k)$. We will in fact *assume* that our space is the total space of a line bundle over $\mathbb{P}^1$ and define the spaces

$$Z_k \coloneqq \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-k)\big) \ ,$$

since we are only interested in the local model: As we will explain in §,2.3, this local model may be glued into a larger space which contains smooth rational curve with normal bundle isomorphic to Z_k. We denote by $\pi\colon Z_k \to X_k$ the contraction of ℓ to a point; the space X_k is the affine cone over the rational normal curve of degree k. In fact, π is a toric resolution given by the inclusion of fans  $\hookrightarrow$  (where the left-most ray has slope $-k$), even though we will never need the toric description.

This chapter is organised as follows. In § 2.2 we describe what holomorphic bundles on Z_k look like. In § 2.3 we provide some background material from mathematical physics which motivates the study of these bundles. That section is largely taken from [GKM]. The study of the moduli proper begins in § 2.4 where we define numerical invariants for bundles. In § 2.5 we state sharp bounds for these invariants, citing mainly from [BGK1], and we tabulate several examples, both for instanton and for non-instanton bundles. In § 2.6 we use the numerical invariants to describe the moduli space of bundles, and in § 2.7 we provide an alternative set of invariants using endomorphism bundles.

2.2 Vector bundles on Z_k

Suppose that $E \to Z_k$ is a holomorphic vector bundle. By the Grothendieck splitting principle, $E|_\ell \cong \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$, and $c_1(E) = \sum_i a_i$. It turns out that in fact E is algebraically filtered, that is, made up from iterated algebraic extensions of bundles:

Theorem 2.1 ([Ga97]). *A holomorphic vector bundle $E \to Z_k$ of rank r is algebraically filtered, i.e. there exists an increasing filtration $E_1 \subset \cdots \subset E_r = E$ such that E_1 is a line bundle and $E_i \big/ E_{i-1}$ is a line bundle for $2 \le i \le r$, and moreover all bundles E_i are algebraic.*

The spaces Z_k are special model spaces, and in fact this result works in much greater generality.

Theorem 2.2 ([BGK2, Section 3]). *Let W be a connected, complex manifold and $\ell \subset W$ a reduced, connected curve that is locally a complete intersection. If the conormal bundle $N^*_{\ell,W}$ is ample, then every vector bundle on $\widehat{\ell}$ is filtrable. That is, if $E \to \widehat{\ell}$ is a vector bundle, then there exists a filtration $E_1 \subset \cdots \subset E_r = E$ such that $E_1, E_i \big/ E_{i-1} \in \mathrm{Pic}(\widehat{\ell})$, $2 \le i \le r$. If in addition ℓ is smooth, then every holomorphic bundle on $\widehat{\ell}$ is algebraic, that is, given by a polynomial rather than a formal power series.*

Proof. An analogous version of this theorem for threefolds will appear in Chapter 3. Since the proof is very similar, we refer the reader to Theorem 3.11 (page 63) for the proof of filtrability.

Algebraicity follows from the fact that every bundle E on the formal space $\widehat{\ell}$ is in fact already determined on some finite infinitesimal neighbourhood $\ell^{(n)}$, that is, the restriction $E \to E|_{\ell^{(n)}}$ is an isomorphism:

Assume now that ℓ is smooth. We know that E is an extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow F \longrightarrow 0$$

and thus classified by $\mathrm{Ext}^1_{\widehat{\ell}}(F, L) = H^1(\widehat{\ell}; F \otimes L^\vee)$. The latter is finite-dimensional, which implies that E is already determined on some finite infinitesimal neighbourhood $\ell^{(n)}$. But $\ell^{(n)}$ is a projective scheme, so E is algebraic by the classical GAGA correspondence. □

Now we specialise to the case of rank-2 bundles. First note that $\mathrm{Pic}\, Z_k \cong H^2(Z_k; \mathbb{Z}) \cong \mathbb{Z}$ and thus line bundles on Z_k are uniquely determined by their first Chern class, and they are simply the pull-back of $\mathcal{O}_{\mathbb{P}^1}(r)$ from $\mathbb{P}^1$ for $r \in \mathbb{Z}$. Now if $E \to Z_k$ is a bundle of rank 2 with $c_1(E) = 0$, then by Grothendieck's splitting principle again, $E|_\ell \cong \mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$, and we call the integer j the *splitting type* of E. Now by Theorem 2.1, E has a filtration $E_1 \subset E_2 = E$ where E_1 and E/E_1

are line bundles. That is, E fits into a short exact sequence

$$0 \longrightarrow \mathcal{O}(-j) \longrightarrow E \longrightarrow \mathcal{O}(j) \longrightarrow 0\ . \tag{2.1}$$

We will also fix once and for all local coordinate charts on Z_k. Since Z_k is the total space of a vector bundle over the Riemann sphere $\mathbb{P}^1$, we only need two charts: Let $U \cong \mathbb{C}^2 = \{z, u\}$ and $V \cong \mathbb{C}^2 = \{z^{-1}, z^k u\}$. The bundle E is thus uniquely determined by one transition function on the overlap $U \cap V$, which can be expressed in the form

$$T = \begin{pmatrix} z^j & p(z,u) \\ 0 & z^{-j} \end{pmatrix}\ , \tag{2.2}$$

where p is a polynomial in z, z^{-1} and u.

Definition 2.3. If $j \in \mathbb{N}$ and $p \in \mathbb{C}[z^{\pm 1}, u]$, we write $E(j, p)$ for the bundle of splitting type j given by the transition function T in Equation 2.2.

Remark 2.4. Similar computations for the case Z_1 (but for *framed* bundles) appear in Section 6 of [Ga08]. In the language of that paper, bundles over Z_k are considered as bundles over the Hirzebruch surface

$$\Sigma_k = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}\big)$$

that are trivial over the line at infinity (and moreover, a choice of trivialisation is fixed).

Our aim is to describe the moduli of all holomorphic rank-2 vector bundles on Z_k with $c_1 = 0$ up to isomorphism. Bundles of higher rank or of different topological type (i.e. different first Chern class) should not pose any greater difficulty in principle, but the computational methods that we employ are much simpler in the case of rank 2 and $c_1 = 0$, where we can describe each bundle conveniently by a single polynomial p and a single integer j.

2.3 Application to mathematical physics

This section illustrates an application of our study of moduli spaces of vector bundles to mathematical physics and the theory of instantons, although it is only tangential to the remainder of this thesis. This section draws heavily from [GKM].

The complex dimension 2 is particular since a complex 2-manifold is also a real 4-manifold, and the geometry of real 4-manifolds is famously very special. The connection between holomorphic vector bundles on complex (compact, Kähler) 2-manifolds and complex vector bundles on real 4-manifolds comes from the *Kobayashi-Hitchin correspondence* between holomorphic bundles and instantons. We begin by introducing instantons.

2.3.1 Instantons

If M^n is a Riemannian manifold of dimension n and $E \to M$ is a smooth vector bundle, then a connection $\nabla\colon \Gamma(E) \to \Gamma(E \otimes T^*M)$ determines an $\mathrm{End}(E)$-valued 2-form F_∇, the *curvature* of the connection. Using the Killing form on the Lie algebra $\mathfrak{g}$ of the structure group of E (e.g. $\mathfrak{so}(k;\mathbb{R})$ for a general rank-k bundle or $\mathfrak{su}(k)$ for a complex rank-k bundle) and the Hodge star on M, we have a a pointwise inner product on each space $\mathrm{End}\, E_x \otimes \Omega^*_x M$, $x \in M$, given by

$$\|\alpha_x\|^2 := \|\alpha_x \wedge *\alpha_x\|^2_{\mathfrak{g}}\,, \text{ or } \|\alpha_x\| = \mathrm{Tr}(\alpha_x \wedge *\alpha_x)\,.$$

If $\|F_\nabla\|$ is integrable (e.g. if M is compact or F_∇ is of class L^2), we can define the *Yang-Mills functional*

$$S_{\mathrm{YM}} := \int_M \|F_\nabla\|^2\,.$$

Applying the variational principle to find the stationary points of S_{YM} gives the equations of motion for a gauge theory on M with values in E, whose solutions are connections on E. The global minima of S_{YM} are called *instantons*.

When $\dim(M) = 2$, instantons are flat connections on E. When $\dim(M) = 4$, instantons are *(anti-)self-dual* connections, i.e. connections whose curvature satisfies $*F_\nabla = \pm F_\nabla$. For mathematicians, instantons on 4-manifolds are usually synonymous with (anti-)self-dual connections. An instanton on a 4-manifold has a *charge* which is an integer and defined as

$$k = \frac{1}{4\pi^2}\int_X \|F_\nabla\|^2\,.$$

The fact that this is indeed an integer follows easily by observing that k is nothing but the second Chern class of E, which lives in $H^2(M;\mathbb{Z}) \cong \mathbb{Z}$.

2.3.2 The Kobayashi-Hitchin correspondence

If the 4-manifold M is also a complex surface, then a connection ∇ on a complex bundle $E \to M$ breaks up into $\nabla = \nabla^{1,0} + \nabla^{0,1}$. If E is also a holomorphic vector bundle, then we say that a connection ∇ is compatible with the holomorphic structure if $\nabla^{0,1} = \overline{\partial}_E$.

The Kobayashi-Hitchin correspondence relates instanton connections and holomorphic structures. Roughly speaking, if X is a complex manifold and $E \to X$ is a complex vector bundle with an instanton connection ∇, then $\overline{\partial}_E := \nabla^{0,1}$ determines a holomorphic structure on E, and conversely a holomorphic bundle E admits an anti-self-dual connection whose $(0,1)$-part is given by $\overline{\partial}_E$.

Restricting ourselves to bundles of rank 2 (that is, $SU(2)$-bundles on the smooth side, and $SL(2;\mathbb{C})$-bundles on the holomorphic side), the precise correspondence is:

$$\left\{ \begin{array}{c} \text{irreducible } SU(2)\text{-in-} \\ \text{stantons } E \text{ of charge } k \end{array} \right\} \overset{\text{K.–H.}}{\Longleftrightarrow} \left\{ \begin{array}{c} \text{stable } SL(2)\text{-bundles} \\ E \text{ with } c_2(E) = k \end{array} \right\} ,$$
$$\nabla = \nabla^{1,0} + \nabla^{0,1} \longleftrightarrow \nabla^{0,1} = \overline{\partial}_E \ . \tag{2.3}$$

A few points require explanation. First off, we observe that instanton bundles are topologically constrained to satisfy $c_1(E) = 0$, hence the 'special' groups. Next, "irreducible" means that the holonomy of the connection is the full group $SU(2)$. Finally, "stable" on the holomorphic side refers to slope stability. That is, E is *slope-stable* if for every proper subbundle $F < E$, $\mu(F) < \mu(E)$, where the *slope* of a bundle is defined as $\mu(E) = \deg(E)/\operatorname{rk}(E)$.

The correspondence has been proved in the cases when X is a projective surface by Donaldson and when X is compact Kähler by Uhlenbeck and Yau. In the non-compact case Donaldson proved the correspondence for $X = \mathbb{C}^2 = Z_0$ and King for the case where X is the blow-up of $\mathbb{C}^2$ at the origin, which we denote by Z_1. In the non-compact cases, an instanton on X has to be understood as an instanton on the projective closure of X, which is $\mathbb{C}P^2$ in the case of $X = \mathbb{C}^2$ and the first Hirzebruch surface Σ_1 when $X = Z_1$, with the additional condition that the bundle be trivial on a neighbourhood of the line at infinity. Of course in the non-compact case the second Chern class and the instanton charge vanish.

2.3.3 Instantons on Z_k: Local models and decay

With this in mind it turns out that a holomorphic rank-2 bundle E on Z_k corresponds to an *instanton* if $c_1(E) = 0$ and if E extends to a bundle on the Hirzebruch surface Σ_k such that the extension is trivial on the complement of ℓ. Because of this condition, not all bundles with $c_1 = 0$ correspond to instantons. The necessary and sufficient condition is the following:

Theorem 2.5 ([GKM, Corollary 5.5]). *A holomorphic rank-2 bundle E on Z_k with $c_1(E) = 0$ is an $SU(2)$-instanton if and only if $E|_\ell \cong \mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$ and $j = nk$ for some integer n.*

The following ad-hoc definition turns out to be useful.

Definition 2.6. A rank-2 bundle E on Z_k is called *framed-stable* if $E|_{Z_k \smallsetminus \ell}$ is trivial and framed on $Z_k \smallsetminus \ell$.

In the next section, we will describe numerical invariants for bundles on Z_k, from which we build the *local holomorphic Euler characteristic* $\chi^{\mathrm{loc}}(E) \in \mathbb{Z}_{\geq 0}$. For now, all we need to know is that if $\pi\colon Z_k \to X_k$ is the contraction of the zero section and if E is a rank-2 bundle on Z_k, then (cf. [Bl96, p. 30])

$$\chi\big(X_k; (\pi_* E)^{\vee\vee}\big) = \chi(Z_k; E) + \chi^{\mathrm{loc}}(E) \ .$$

In [GKM] a version of the Kobayashi-Hitchin correspondence was proved for the spaces Z_k, which relates $SU(2)$-instantons on Z_k to framed-stable holomorphic rank-2 bundles with vanishing first Chern class and local holomorphic Euler characteristic n ([GKM, Prop. 5.3]):

$$\left\{ \begin{array}{c} SU(2)\text{-instantons on} \\ Z_k \text{ with local charge } n \end{array} \right\} \overset{\text{K.-H.}}{\Longleftrightarrow} \left\{ \begin{array}{c} \text{framed-stable } SL(2;\mathbb{C})\text{-} \\ \text{bundles on } Z_k \text{ with } \chi^{\mathrm{loc}} = n \end{array} \right\} . \tag{2.4}$$

In this correspondence, we use the local holomorphic Euler characteristic to *define* the local charge on the instanton side.

We will now explain how the spaces Z_k provide a local model for instanton decay. This involves a process that was called "holomorphic surgery" in [GKM].

To this end, let Z be any compact complex surface that contains an embedded smooth rational curve $\ell \subset Z$ with $\ell^2 = -k$, that is, $N_{\ell/Z} \cong Z_k$. Let us write $U(\ell)$ for a small analytic neighbourhood of ℓ. We have $Z = (Z \smallsetminus \ell) \cup U(\ell)$. Let us fix some notation:

$$Z^\circ := U(\ell) \smallsetminus \ell = (Z \smallsetminus \ell) \cap U(\ell) \qquad\qquad Z_k^\circ := Z_k \smallsetminus \ell$$

It will become clear presently that if E is an instanton bundle on Z, then we require that $E|_{Z^\circ}$ be holomorphically trivial.

Definition 2.7. We say that two bundles E_1, E_2 on Z are related by *holomorphic surgery around ℓ* if $E_1|_{Z \smallsetminus \ell} \cong E_2|_{Z \smallsetminus \ell}$. Furthermore, if $c_2(E_1) > c_2(E_2)$, we say that E_1 *decays to* E_2.

The key idea is that all the data describing holomorphic surgery around ℓ is captured by a bundle on Z_k. Note that unlike in the smooth category, it is not true in general that $U(\ell)$ is biholomorphic to Z_k, but it follows from [Ga97] that bundles on Z_k and on $U(\ell)$ are already determined on a finite infinitesimal neighbourhood of ℓ, and so for the purpose of holomorphic surgery and instanton decay, $U(\ell)$ and Z_k can be identified.

Now we use the requirement that instanton bundles be trivial on Z° and Z_k°, respectively, and we fix framings.

Definition 2.8. Let $p\colon E \to Z$ be an instanton bundle. We say that two pairs

$$f = (f_1, f_2)\colon Z^\circ \to p^{-1}(Z^\circ) \qquad \text{and} \qquad g = (g_1, g_2)\colon Z^\circ \to p^{-1}(Z^\circ)$$

of fibrewise linearly independent holomorphic sections of $E|_{Z^\circ}$ are *equivalent*, written $f \sim g$, if $\phi := g \circ f^{-1}\colon E|_{Z^\circ} \to E|_{Z^\circ}$ extends to a holomorphic map $\phi\colon E \to E$ over all of Z. A *frame* of E over Z° is an equivalence class of fibrewise linearly independent holomorphic sections of E over Z°.

We define a frame over Z_k° analogously. Also, if $\varpi\colon Z \to X$ is the contraction of ℓ to $x \in X$, we define a frame of a vector bundle on X over $U(x) \smallsetminus \{x\}$, where $U(x)$ is a small disk neighbourhood of x. Thus we have three notions of framed bundles:

- A *framed bundle* $\overline{E}^f$ on Z is a pair consisting of a bundle $\overline{E} \to Z$ together with a frame of $\overline{E}$ over Z°.
- A *framed bundle* V^f on Z_k is a pair consisting of a bundle $V \to Z_k$ together with a frame of V over Z_k°.
- A *framed bundle* E^f on X is a pair consisting of a bundle $E \to X$ together with a frame of E over $U(x) \smallsetminus \{x\}$. We will always take $U(x) = \varpi\big(U(\ell)\big)$.

The power of these definitions lies in the fact that we can now describe any framed bundle on Z in terms of a framed bundle on X and a framed bundle on Z_k. We have separated the information near ℓ from the information away from ℓ.

Proposition 2.9. *An isomorphism class* $[\overline{E}^f]$ *of a framed bundle on* Z *is uniquely determined by a pair of isomorphism classes of framed bundles* $[E^f]$ *on* X *and* $[V^f]$ *on* Z_k*. We write* $\overline{E}^f = (E^f, V^f)$.

Proof. The important step is to set $E := (\varpi_* \overline{E})^{\vee\vee}$. This is a reflexive sheaf on X and by construction trivial on $U(x) \smallsetminus \{x\}$. Since a reflexive sheaf is determined on the complement of a point (e.g. see [Har80, Prop. 1.6]), E is trivial on $U(x)$ and completely determined on $X \smallsetminus \{x\}$. The remainder of the proof consists of checking the definitions, for which we refer to [GKM, Prop. 3.4]. $\square$

We have now established how to use our model spaces Z_k to describe the local information of a vector bundle $E \to Z$ on a space Z that contains a curve ℓ with self-intersection number $-k$. We can thus study the purely local situation on Z_k. The following result is [GKM, Prop. 4.1].

Proposition 2.10. *Let E_1 and E_2 be $SL(2;\mathbb{C})$-bundles over Z_k with splitting types j_1 and j_2, respectively. There exists an isomorphism $E_1|_{Z_k^\circ} \cong E_2|_{Z_k^\circ}$ if and only if $j_1 \equiv j_2 \mod k$. In particular, E_1 can decay totally over Z_k (that is, be related by holomorphic surgery to the trivial bundle) if and only if $j_1 \equiv 0 \mod k$.*

This proposition proves Theorem 2.5: Since we require instantons on Z_k to be bundles that are trivial away from ℓ, we must have $j \equiv 0 \pmod k$.

Let us restate how the local holomorphic Euler characteristic comes in: If $\overline{E}_i^f = (E_i^f, V_i^f)$, $i = 1, 2$, are two framed bundles on a general space Z as before that are related by holomorphic surgery around ℓ, then Proposition 2.9 shows that $E_1 \cong E_2$, and so all the information about the holomorphic surgery is encoded in the bundles V_i on Z_k. In particular, the difference of instanton charges is precisely $\chi^{\mathrm{loc}}(V_1) - \chi^{\mathrm{loc}}(V_2)$.

From the work in the subsequent sections, we shall obtain existence and nonexistence results for instantons on Z_k. For example, we find that there are no instantons with $0 < \chi^{\mathrm{loc}} \leq k-2$ on Z_k when $k > 2$, and for $k \geq 2$ there exists $(k-2)$-dimensional families of (unframed) instantons with $\chi^{\mathrm{loc}} = k-1$ on Z_k.

Let us close this section by stating that the author is currently ignorant of any physical meaning of χ^{loc} in the non-instanton case.

2.4 Numerical invariants

To study bundles on Z_k, we consider the action of the contraction $\pi\colon Z_k \to X_k$ of the zero section ℓ. To motivate this, let us return for a moment to the case of instantons on Z_1 and Σ_1. Here the contraction maps Z_1 to $\mathbb{C}^2$, and Σ_1 to $\mathbb{P}^2$. Since the target is smooth, the direct image sheaf $\pi_* E$ of an instanton E is a sum of a locally free sheaf and torsion, so its double dual $(\pi_* E)^{\vee\vee}$ is locally free. On the compact spaces we can thus consider the difference $c_2^{\mathrm{loc}}(E) \coloneqq c_2\big(E\big) - c_2\big((\pi_* E)^{\vee\vee}\big)$.

This is the local charge of E which appeared in the discussion in § 2.3, which we recall: If X is a smooth, compact surface with an embedded $C \subset X$ such that $C \cong \mathbb{P}^1$ and $N_{C/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\widetilde{E}, \widetilde{F}$ are instantons on X satisfying $\widetilde{E}|_{X \smallsetminus C} \cong \widetilde{F}|_{X \smallsetminus C}$, then $c_2(\widetilde{E}) - c_2(\widetilde{F}) = c_2^{\mathrm{loc}}(E) - c_2^{\mathrm{loc}}(F)$, where $E, F \to Z_k$ the the bundles on Z_k isomorphic to $\widetilde{E}, \widetilde{F}$ restricted to a neighbourhood of C.

We compute $c_2^{\mathrm{loc}}(E)$ directly by an application of Riemann-Roch and find

$$c_2\big(E\big) - c_2\big((\pi_* E)^{\vee\vee}\big) = h^0\big(X; (\pi_* E)^{\vee\vee}/\pi_* E\big) + h^0\big(X; R^1\pi_* E\big)\,. \tag{2.5}$$

The notion of a Chern class of a holomorphic bundle is well-defined on smooth manifolds, but on singular spaces there exist several inequivalent notions of Chern classes. However, the right-hand side of Equation 2.5 is independent of any notion of Chern class. In fact, it is a special case of what Blache [Bl96] defines as the *local holomorphic Euler characteristic* χ^{loc} of a reflexive sheaf near an isolated quotient singularity: Let $\sigma\colon (X, A) \to (X', x)$ be a resolution of an isolated quotient singularity and $\mathcal{F}$ a reflexive sheaf on X. Then

$$\chi^{\text{loc}}\big(A, \mathcal{F}, \sigma\big) := h^0\big(X'; (\sigma_*\mathcal{F})^{\vee\vee}/\sigma_*\mathcal{F}\big) + \sum_{i=1}^{n-1}(-1)^{i-1}h^0\big(X'; R^i\sigma_*\mathcal{F}\big) . \tag{2.6}$$

For the case when X' is an orbifold, Blache [Bl96] shows that,

$$\chi\big(X', (\sigma_*\mathcal{F})^{\vee\vee}\big) = \chi\big(X, \mathcal{F}\big) + \sum_{x\in\operatorname{Sing} X'} \chi^{\text{loc}}\big(\sigma^{-1}(x), \mathcal{F}, \sigma\big) ,$$

so the local holomorphic Euler characteristic measures precisely the amount of total Euler characteristic that is gained by contracting the orbifold resolution, or in other words the contribution from a neighbourhood of the exceptional set A.

Our spaces Z_k have cohomological dimension 1, so all higher derived images $R^i\pi_*E$ vanish for $i > 1$. For the smooth case Z_1 we have thus

$$c_2^{\text{loc}}\big(\ell, E\big) = \chi^{\text{loc}}\big(\ell, E, \pi\big) ,$$

and from here we *define* the *local charge of E near ℓ* to be $\chi^{\text{loc}}\big(\ell, E, \pi\big)$. We name the two constituent summands the *width* $w_k(E)$ and the *height* $h_k(E)$ of the bundle E,

Definition 2.11.

$$\chi^{\text{loc}}\big(\ell, E, \pi\big) = h^0\big(X_k; (\pi_*E)^{\vee\vee}/\pi_*E\big) + h^0\big(X_k; R^1\pi_*E\big) = w_k(E) + h_k(E) ,$$

i.e.

$$w_k(E) \;:=\; h^0\big(X_k; (\pi_*E)^{\vee\vee}/\pi_*E\big) \text{ and} \tag{2.7}$$

$$h_k(E) \;:=\; h^0\big(X_k; R^1\pi_*E\big) . \tag{2.8}$$

Remark 2.12. The width $w_k(E)$ measures how far the direct image sheaf π_*E is from being reflexive; the height $h_k(E)$ measures how close E is to being the split bundle (which is the unique bundle with maximal $w_k(E) + h_k(E)$ for a fixed j).

The computation of the width and the height makes use of the Theorem on Formal Functions (Theorem 1.6):

$$(R^i\pi_*E)_0^\wedge \cong \varprojlim_n H^i(\ell^{(n)}; E|_{\ell^{(n)}})$$

The left-hand side is the completion of the stalk downstairs on X_k which we are interested in, and the theorem tells us that we can compute it by computing cohomology upstairs on Z_k and taking the inverse limit over the infinitesimal neighbourhoods of the exceptional set. The fact that we can only compute the *completion* on the left-hand side is not a problem, since we are only interested in dimensions of spaces of global sections, and dimension is invariant under completion.

Remark 2.13. In the preceding definitions, it does not matter whether one works with algebraic sheaves or their analytification, and whether one works with the Zariski topology or the analytic one. The completion of a polynomial ring and its associated ring of convergent power series is the same.

In the sequel we will write $\chi(\ell, E)$ in place of $\chi^{\text{loc}}(\ell, E, \pi)$, there should be no ambiguity.

2.5 Bounds on the numerical invariants

The following results were proved in [BGK1].

Theorem 2.14. *Let E be a rank-2 bundle over Z_k of splitting type j. Then the following bounds are sharp: For $j > 0$ and with $n_2 = \lfloor \frac{j}{k} \rfloor$,*

$$0 \le w_k(E) \le (j+1)n_2 - kn_2(n_2+1)/2 \qquad \textit{for } k > 1,$$

and

$$1 \le w_1(E) \le j(j+1)/2 \, .$$

Furthermore, for all $0 < j < k$, $w_k(E) = 0$ for all bundles E (and necessarily $k > 1$).

Proposition 2.15. *Let $E(j,p)$ be the bundle of splitting type j whose extension class is given by p, and let $\bar{E}(j) := \mathcal{O}(-j) \oplus \mathcal{O}(j)$ denote the split bundle. If $u | p(z,u)$ and $p \not\equiv 0$, then*

$$h_k\big(\bar{E}(j)\big) \ge h_k\big(E(j,p)\big) \, .$$

Theorem 2.16. *Let E be a rank-2 bundle over Z_k of splitting type $j > 0$. Set $n_1 = \left\lfloor \frac{j-2}{k} \right\rfloor$. The following bounds are sharp:*

$$j - 1 \leq h_k(E) \leq (j-1)(n_1+1) - k(n_1+1)n_1/2 \,.$$

Corollary 2.17. *Let E be a rank-2 bundle over Z_k of splitting type j with $j > 0$ and let $j = nk + b$ as above. The following are sharp bounds for the local holomorphic Euler characteristic of E:*

$$j - 1 \leq \chi(\ell, E) \leq \begin{cases} n^2k + 2nb + b - 1 & \text{if } k \geq 2 \text{ and } 1 \leq b < k \,, \\ n^2 k & \text{if } k \geq 2 \text{ and } b = 0 \,, \end{cases}$$

and

$$j \leq \chi(\ell, E) \leq j^2 \text{ for } k = 1 \,.$$

Instanton existence and charge gaps. Since the bounds on $\chi(\ell, E)$ are sharp, we can deduce the existence and non-existence of instantons, i.e. bundles with splitting type $j = nk$ (cf. Theorem 2.5) with prescribed local charges. We see that for $k = 1$, all natural numbers occur as local instanton charges of some bundle, but for $k > 2$, the local charges $1, \ldots, k-2$ can *never* occur. If one requires instantons of local charge 1 for instanton decay via holomorphic surgery, one would say that self-intersection number of ℓ is an obstruction to instanton decay.

Examples of numerical invariants. Before we move on to study moduli spaces, we tabulate the values of the numerical invariants w_k and h_k for bundles $E(j, p)$ on Z_k for several values of j and k. We also include at this point two values h^1 and h^0 which are treated in the final Section 2.7; see Definition 2.29.

We treat the case of instanton bundles, for which $j = nk$ for some n, separately from the case of non-instanton bundles, for which $j \not\equiv 0 \pmod{k}$. The example values are listed in Table 2.1.

2.6 Moduli

We would like to know the structure on the space of rank-2 bundles on Z_k. We already know from Theorem 2.1 that all such bundles are extensions of the form (2.1). For each fixed splitting type j, the space of such extensions is[1] $\operatorname{Ext}^1_{Z_k}\big(\mathcal{O}(j), \mathcal{O}(-j)\big) \cong H^1\big(Z_k; \mathcal{O}(-2j)\big)$.

[1] The isomorphism is, for any locally free $\mathcal{O}$-module $\mathcal{L}$ of finite rank (see [Har77, Props. 6.7 and 6.3]),

$$\operatorname{Ext}^i_{\mathcal{O}}(\mathcal{L}, -) := R^i \operatorname{Hom}_{\mathcal{O}}(\mathcal{L}, -) \cong R^i \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}, - \otimes \mathcal{L}^\vee) \cong R^i\Gamma(- \otimes \mathcal{L}^\vee) =: H^i(- \otimes \mathcal{L}^\vee) \,.$$

k	j	p	(w_k, h_k)	h^1	h^0
1	2	u, zu	$(1,1)$	4	2
1	2	zu^2	$(2,1)$	5	1
1	2	0	$(3,1)$	6	0
1	3	$z^{-1}u, z^2u$	$(3,2)$	11	4
1	3	u, zu	$(1,2)$	9	6
1	3	$z^{-1}u + z^2u$	$(1,2)$	9	6
1	3	u^2, z^2u^2	$(3,3)$	12	3
1	3	zu^2	$(2,3)$	11	4
1	3	$u^2 + z^2u^2$	$(2,3)$	11	4
1	3	zu^3, z^2u^3	$(4,3)$	13	2
1	3	z^2u^4	$(5,3)$	14	1
1	3	0	$(6,3)$	15	0
2	6	$z^{-3}u, z^5u$	$(6,7)$	31	5
2	6	$z^{-2}u, z^4u$	$(4,6)$	28	8
2	6	$z^{-1}u, z^3u$	$(2,5)$	25	11
2	6	u, z^2u	$(1,5)$	24	12
2	6	zu	$(0,5)$	23	13
2	6	$z^{-3}u + z^2u$	$(0,5)$	23	13
2	6	$z^{-2}u + z^3u$	$(0,5)$	23	13
2	6	$z^{-1}u + z^4u$	$(0,5)$	23	13
2	6	$u + z^2u$	$(1,5)$	24	12
2	6	$z^{-1}u + z^3u$	$(1,5)$	24	12
2	6	$z^{-2}u + z^4u$	$(1,5)$	24	12
2	6	$z^{-1}u^2, z^5u^2$	$(6,8)$	32	4
2	6	u^2, z^4u^2	$(4,8)$	30	6
2	6	$z^{\{1,2,3\}}u^2$	$(2,8)$	28	8
2	6	$u^2 + z^4u^2$	$(2,8)$	28	8
2	6	$\ldots u^3$			
2	6	$z^{\{3,4,5\}}u^4$	$(7,9)$	34	2
2	6	z^5u^5	$(8,9)$	35	1
2	6	0	$(9,9)$	36	0
3	3	zu	$(0,2)$	6	1
3	3	0	$(1,2)$	7	0
3	6	zu	$(0,5)$	19	7
3	6	0	$(5,7)$	26	0
3	9	zu	$(0,8)$	38	19
3	9	0	$(12,15)$	57	0
4	4	zu	$(0,3)$	9	1
4	4	0	$(1,3)$	10	0
4	8	zu	$(0,7)$	27	9
4	8	0	$(6,10)$	36	0
4	12	zu	$(0,11)$	53	25
4	12	0	$(15,21)$	78	0

k	j	p	(w_k, h_k)	h^1	h^0
2	3	u	$(1,2)$	7	2
2	3	zu	$(0,2)$	7	2
2	3	z^2u	$(1,2)$	7	2
2	3	$u + z^2u$	$(0,2)$	7	2
2	3	z^2u^2	$(2,2)$	8	1
2	3	0	$(2,2)$	9	0
3	4	u	$(1,3)$	10	2
3	4	zu, z^2u	$(0,3)$	10	2
3	4	z^3u	$(1,3)$	10	2
3	4	$u + z^3u$	$(0,3)$	10	2
3	4	z^3u^2	$(2,3)$	11	1
3	4	0	$(2,3)$	12	0
3	5	$z^{-1}u$	$(2,4)$	16	2
3	5	u	$(1,4)$	15	3
3	5	zu, z^2u	$(0,4)$	14	4
3	5	z^3u	$(1,4)$	15	3
3	5	z^4u	$(2,4)$	16	2
3	5	$u + z^4u$	$(0,4)$	14	4
3	5	$z^{-1}u + z^4u$	$(1,4)$	15	3
3	5	z^2u^2	$(2,5)$	17	1
3	5	z^3u^2	$(2,5)$	17	1
3	5	z^4u^2	$(2,5)$	17	1
3	5	0	$(3,5)$	18	0

Table 2.1: Numerical invariants of some instanton bundles (left) and non-instanton bundles (right).

Remark 2.18. Note that the space $\mathrm{Ext}^1_{Z_k}\big(\mathcal{O}(j),\mathcal{O}(-j)\big)$ is finite-dimensional for every j. Hence holomorphic bundles on Z_k are *algebraic*. Yet another way to see this is to note that a bundle E is already determined on a *finite* infinitesimal neighbourhood $\ell^{(m)}$, which is a projective scheme and thus automatically satisfies GAGA.

We will introduce our moduli space of vector bundles step by step. To begin, let us write $\mathfrak{M}(Z_k)$ for the *set* of isomorphism classes of all holomorphic vector bundles E on Z_k with rank 2 and $c_1(E) = 0$. More generally, we will use this definition of $\mathfrak{M}(X)$ whenever $X = \mathrm{Tot}\Big(\bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)\Big)$ is the total space of a sum of line bundles on $\mathbb{P}^1$. In the next chapter we will encounter the analogous setup for certain threefolds of this kind.

The first observation is that the restriction of a bundle E to the zero section $\ell \subset Z_k$ (or more generally $\ell \in X$) determines the splitting type $j \in \mathbb{Z}_{\geq 0}$, where $E|_\ell \cong \mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$. Since bundle isomorphisms preserve the splitting type, it is clear that $\mathfrak{M}(Z_k)$ breaks up into a disjoint union of sets

$$\mathfrak{M}(Z_k) = \bigsqcup_{j=0}^{\infty} \mathfrak{M}(Z_k; j) \;,$$

where $\mathfrak{M}(Z_k; j)$ denotes the set of isomorphism classes of bundles of splitting type j.

We can immediately endow these pieces with a topology: The discussion above shows bundles of splitting type j are parametrised by the finite-dimensional, complex vector space $\mathrm{Ext}^1_{Z_k}\big(\mathcal{O}(j),\mathcal{O}(-j)\big)$. Writing $\sim$ for the equivalence relation of bundle isomorphism, we give $\mathfrak{M}(Z_k; j)$ the quotient topology

$$\mathfrak{M}(Z_k; j) = \mathrm{Ext}^1_{Z_k}\big(\mathcal{O}(j),\mathcal{O}(-j)\big)\big/\!\sim \;.$$

We will show later that the group of bundle isomorphisms is not in general reductive and that the quotient topology is rather unwieldy. For example, fix a non-split bundle $E(j,p)$ of splitting type j (for example, $p(z,u) = zu$), and consider the family $E(j,\epsilon p)$ for $\epsilon \to 0$. This shows that the split bundle $E(j,0)$ can be deformed into any bundle of the same splitting type, and whatever structure $\mathfrak{M}(Z_k; j)$ has, the split bundle is "near" every other bundle. (Later we will address this by relegating the split bundle to a separate stratum.) Nevertheless, these pieces give us an explicit handle. Part of this thesis consists of providing a means to decompose these spaces into finer components that possess a richer structure, for example a "generic" subset which possesses the structure of a manifold (or even a quasi-projective variety).

Note moreover that we are not topologising the entire moduli set $\mathfrak{M}(Z_k)$. If we were to attempt this, the components $\mathfrak{M}(Z_k; j)$ would have to get "very close" to each other at several points: Consider for example the family $E(j, zu + \epsilon z)$. For $\epsilon = 0$, this lies in $\mathfrak{M}(Z_k; j)$, but for non-zero values of ϵ, this bundle[2] has splitting type $< j$. We avoid this complication by always decomposing the moduli set into the pieces of fixed splitting type.

After choosing coordinates, we can compute the space of extensions explicitly and find that it can be described as the space of coefficients of the polynomial p that appears in the transition function T in (2.2) for the bundle E.

Proposition 2.19. *Let E be a holomorphic vector bundle of rank 2 on Z_k. The polynomial p appearing in its transition function* (2.2) *may be chosen to be of the form*

$$p(z, u) = \sum_{r=1}^{\lfloor (2j-2)/k \rfloor} \sum_{s=kr-j+1}^{j-1} p_{rs} \, z^s u^r \, . \tag{2.9}$$

Proof. The proof is a computation that is entirely analogous to the corresponding statement on threefolds in Proposition 3.1 in the next section; please refer to that proof for details, or alternatively see [Ga97, Theorem 3.3]. □

We say that p is in *canonical form* if it is as in Proposition 2.19. Note that in this form, p is always divisible by u, which means that the restriction of E to ℓ splits as $\mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$. We may also consider the slightly more general form

$$\widetilde{p}(z, u) = \sum_{r=0}^{\lfloor (2j-2)/k \rfloor} \sum_{s=kr-j+1}^{j-1} p_{rs} \, z^s u^r \, ,$$

but note that when p is not divisible by u, then the splitting type of E is in fact lower than j (see the footnote for an example). This slightly more general notion will be useful from the point of view of deformation theory below.

From the explicit description of p we see that a bundle E splits on the m^{th} infinitesimal neighbourhood $\ell^{(m)}$ if and only if p is a multiple of u^{m+1}. It is rather special for a bundle to split on the first neighbourhood, and bundles that already fail to split on the first neighbourhood are the most important case that we shall consider. On the first neighbourhood, we have a result.

[2] Here is a simpler example how a bundle $E(j, p)$ has a splitting type lower than j when $u \nmid p$. We show that $E(2, 1)$ actually the trivial bundle: $\begin{pmatrix} 1 & 0 \\ z^{-2}-1 & -1 \end{pmatrix} \begin{pmatrix} z^2 & 1 \\ 0 & z^{-2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1-z^2 & -z^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Note that the matrix on the left is holomorphic on the V-chart and the matrix on the right is holomorphic on the U-chart.

The results in this section were published in a joint paper with my supervisor and with E. Ballico [BGK1]. We reproduce them here more or less verbatim.

Proposition 2.20. *If p and p' are two polynomials determining respectively two bundles E and E' on Z_k of splitting type j, then $E|_{\ell^{(1)}} \cong E'|_{\ell^{(1)}}$ if and only if $p = \lambda p'$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.*

Proof (cf. [BGK1, Theorem 4.9]). The "if" part is clear. Now suppose $E|_{\ell^{(1)}}$ and $E'|_{\ell^{(1)}}$ are isomorphic. We write p_1 and p_1', respectively, for the polynomials determining $E|_{\ell^{(1)}}$ and $E'|_{\ell^{(1)}}$; i.e. the reduction modulo u^2 of p and p'. We have $p_1 = \sum_{s=k-j+1}^{j-1} p_{1s}\, z^s\, u$ and $p_1' = \sum_{s=k-j+1}^{j-1} p_{1s}'\, z^l\, u$. We will write the isomorphism in the form

$$\begin{pmatrix} z^j & p_1' \\ 0 & z^{-j} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} z^j & p_1 \\ 0 & z^{-j} \end{pmatrix},$$

where a, b, c and d are holomorphic on U, and α, β, γ and δ are holomorphic on V. On the first infinitesimal neighbourhood, this yields the following set of equations:

$$\begin{cases} \big(a_0(z) + a_1(z)u\big)z^j + p_1' c_0(z) = \big(\alpha_0(z^{-1}) + \alpha_1(z^{-1})zu\big)z^j & \text{(A1)} \\ z^{-j}\big(c_0(z) + c_1(z)u\big) = \big(\gamma_0(z^{-1}) + \gamma_1(z^{-1})zu\big)z^j & \text{(A2)} \\ \big(b_0(z) + b_1(z)u\big)z^j + p_1' d_0(z) = \alpha_0(z^{-1})p_1 + \big(\beta_0(z^{-1}) + \beta_1(z^{-1})zu\big)z^{-j} & \text{(A3)} \\ z^{-j}\big(d_0(z) + d_1(z)u\big) = \gamma_0(z^{-1})p_1 + \big(\delta_0(z^{-1}) + \delta_1(z^{-1})zu\big)z^{-j} & \text{(A4)} \end{cases}$$

Recalling that p_1 and p_1' are multiples of u and equating terms that are independent of u in (A1) and (A4) gives $a_0(z) = \alpha_0(z^{-1})$ and $d_0(z) = \delta_0(z^{-1})$ respectively. Therefore a_0, α_0, d_0 and δ_0 are constants, and $a_0 = \alpha_0$ and $d_0 = \delta_0$. Next we equate terms in u in Equation (A3), obtaining

$$b_1(z)u\, z^j + p_1' d_0 = \alpha_0 p_1 + \beta_1(z^{-1})u\, z^{-j}\,.$$

In Equation (A3), $z^j\, b_1$ has only terms z^l for $l \geq j$, and $z^{-j}\,\beta_1$ has only terms z^l for $l \leq -j$. Consequently, they do not affect the terms appearing in p_1 and p_1'; and the remaining part of Equation (A3) gives $p_1' d_0 = \alpha_0 p_1$. We observe that p_1 and p_1' differ by a constant.

It remains to show that d_0 and α_0 are non-zero. Taking terms that are independent of u in Equation (A3) we have $b_0(z)\, z^j = \beta_0(z^{-j})\, z^{-j}$, which implies $b_0(z) = \beta_0(z^{-1}) = 0$. It follows that over the exceptional divisor our coordinate change has determinant $a_0\, d_0$, hence $\alpha_0\, \delta_0 = a_0\, d_0 \neq 0$. □

Making the notion of genericity precise amounts to finding a suitable notion of *stability* on the moduli of bundles. We repeat the definition of $\mathfrak{M}(Z_k; j)$ here, and this time we include a notion of stability.

Definition 2.21. For each integer j and each k we define

$$\mathfrak{M}(Z_k; j) := \operatorname{Ext}^1_{Z_k}\big(\mathcal{O}(j), \mathcal{O}(-j)\big)\big/\!\sim$$

to be the space of extensions of $\mathcal{O}(j)$ by $\mathcal{O}(-j)$ on Z_k up to bundle isomorphism; the topology is the quotient topology. If $E \in \mathfrak{M}(Z_k; j)$ is such an extension, we say that E is (h, w)*-stable* if the width $w_k(E)$ and the height $h_k(E)$ of E attain the minimal value (namely $w_k(E) = 0$ and $h_k(E) = j - 1$ as per Theorems 2.14 and 2.16.

Remark 2.22. The lower bound is always realised on all spaces Z_k by the bundle determined by $p(z, u) = zu$. The upper bound is always realised by the split bundle $p(z, u) = 0$.

Definition 2.23. We say that a bundle $E \in \mathfrak{M}(Z_k; j)$ is *generic* if for any other bundle $E' \in \mathfrak{M}(Z_k; j)$ we have the implication $E|_{\ell^{(1)}} \cong E'|_{\ell^{(1)}} \Rightarrow E \cong E'$, and if E is not isomorphic to the split bundle $\mathcal{O}(-j) \oplus \mathcal{O}(j)$.

Remark 2.24. Definition 2.23 says that generic bundles are precisely those which are determined on $\ell^{(1)}$. The explicit exclusion of the split bundle is for convenience to exclude marginal cases; when $j > k$, there are always non-split bundles that split on $\ell^{(1)}$ and so the exclusion of the split bundle from the set of generic bundles is automatic.

Theorem 2.25. *For $j \geq k$, $\mathfrak{M}(Z_k; j)$ the subset of generic bundles form an open, dense subspace homeomorphic to a complex projective space $\mathbb{P}^{2j-2-k}$ minus a closed subvariety of codimension at least $k + 1$.*

Proof (cf. [BGK1, Theorem 4.11]). Let E and E' be the bundles on Z_k given respectively by the extension classes p and p'. By Proposition 2.20 we know that on $\ell^{(1)}$ the only isomorphism of bundles is scaling, so we may assume

$$p = p_1 + p_2 \qquad \text{and} \qquad p' = p_1 + p_2' ,$$

where $p_1 := p|_{\ell^{(1)}} = p'|_{\ell^{(1)}}$. If E and E' are isomorphic, then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z^{-j} & -p \\ 0 & z^j \end{pmatrix} = \begin{pmatrix} a + z^{-j} p' c & z^{2j} b + z^j (dp' - ap) - cpp' \\ z^{-2j} c & d - z^{-j} pc \end{pmatrix}, \tag{2.10}$$

for some change of coordinates $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ holomorphic on U and $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ holomorphic on V, which we may assume have determinant one. We write

$$\alpha(z^{-1}, z^k u) = \alpha_0(z^{-1}) + \alpha_1(z^{-1})\, z^k u + \cdots, \text{ similarly for } \beta, \gamma, \delta, \text{ and}$$

$$a(z,u) = a_0(z) + a_1(z) u + a_2(z) u^2 + \cdots, \text{ similarly for } b, c, d,$$

where the coefficients are convergent power series in z^{-1} or z, respectively. Then

$$\begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 z^{2j} \\ c_0 z^{-2j} & d_0 \end{pmatrix},$$

implies $\beta_0 = b_0 = 0$, and $c_0(z) = c_{00} + c_{01} z + \cdots + c_{0,2j} z^{2j}$. It follows that $\alpha_{00} = a_{00} = \lambda = \delta_{00}^{-1} = d_{00}^{-1} = 1$.

The coefficients of u are

$$\begin{pmatrix} \alpha_1 z^k u & \beta_1 z^k u \\ \gamma_1 z^k u & \delta_1 z^k u \end{pmatrix} = \begin{pmatrix} a_1 u + p_1 c_0 z^{-j} & b_1 u z^{2j} + z^j (\cancel{d_0 p_1 - a_0 p_1}) \\ c_1 u z^{-2j} & d_1 u - p_1 c_0 z^{-j} \end{pmatrix}, \tag{2.11}$$

which has to be holomorphic in $(z^{-1}, z^k u)$. This forces $b_1(z) = b_{10} + b_{11} z + \cdots + b_{1,-2j+k} z^{-2j+k}$, whence $b_1 \neq 0$ provided $k - 2j \geq 0$, i.e. $j \leq \lfloor \frac{k}{2} \rfloor$. But by assumption, $j \geq k$, so that $b_1 = 0$, and consequently $d_1 = -a_1$. Furthermore, assuming for the moment that $j \geq k+1$, the $(1,1)$-entry of (2.11) entails the following relations between the terms of a_1 and c_0:

$$\begin{pmatrix} a_{1,k+1} \\ a_{1,k+2} \\ \vdots \\ a_{1,2j-2} \\ a_{1,2j-1} \end{pmatrix} + \begin{pmatrix} p_{1,j-1} & p_{1,j-2} & \cdots & p_{1,k-2+2} & p_{k-j+1} \\ 0 & p_{1,j-1} & \cdots & p_{1,k-j+3} & p_{1,k-j+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{1,j-1} & p_{1,j-2} \\ 0 & 0 & \cdots & 0 & p_{1,j-1} \end{pmatrix} \begin{pmatrix} c_{0,k+2} \\ c_{0,k+3} \\ \vdots \\ c_{0,2j-1} \\ c_{0,2j} \end{pmatrix} = 0, \tag{2.12}$$

and $a_{1,s} = 0$ for $s \geq 2j$.

On $\ell^{(2)}$, the terms in (2.10) with u^2 are

$$\begin{pmatrix} \alpha_2 z^{2k} u^2 & \beta_2 z^{2k} u^2 \\ \gamma_2 z^{2k} u^2 & \delta_2 z^{2k} u^2 \end{pmatrix} =$$
$$\begin{pmatrix} a_2 u^2 + z^{-j}(p_2' c_0 + p_1 c_1 u) & b_2 u^2 z^{2j} + z^j\big((d_1 - a_1)\, u\, p_1 + (p_2 - p_2')\big) - c_0\, p_1^2 \\ c_2\, u^2 z^{-2j} & d_2\, u^2 - z^{-j}(p_2\, c_0 + p_1\, c_1\, u) \end{pmatrix}.$$

We need to examine the $(1, 2)$-entry:

$$b_2\, u^2 z^{2j} + z^j\big((d_1 - a_1)\, u\, p_1 + (p_2 - p_2')\big) - c_0\, p_1^2 \tag{2.13}$$

The conditions on the expression (2.13) is that all coefficients of $z^l\, u^2$ vanish for $l \geq 2k$, but any coefficient of $z^{2j}\, u^2$ and higher can be cancelled by choosing b_2 appropriately. So we only need to consider the range $k + 1 \leq l \leq 2j - 1$ to verify when the expression

$$z^j(d_1 - a_1)u\, p_1 + z^j(p_2 - p_2') - c_0\, p_1^2$$

can be made holomorphic on V for any choice of p_2'. Given the determinant-one condition on the coordinate changes, this becomes

$$z^j(-2a_1)u\, p_1 + z^j(p_2 - p_2') - c_0\, p_1^2\,,$$

and plugging in the values of p,

$$-2z^j a_1 \sum_{s=k-j+1}^{j-1} p_{1s}\, z^s\, u + z^j \sum_{s=2k-j+1}^{j-1} \big(p_{2,s} - p_{2,s}'\big) z^s\, u^2 - c_0 \left(\sum_{s=k-j+1}^{j-1} p_{1s}\, z^s\, u \right)^2. \tag{2.14}$$

The terms to be cancelled are:

- Step 1, the coefficient of $z^{2k+1}u^2$:

$$\begin{aligned} &-2\big(a_{1,k}p_{1,k-j+1} + a_{1,k-1}p_{1,k-j+2} + \cdots + a_{1,0}p_{1,2k-j+1}\big) + \big(p_{2,2k-j+1} - p_{2,2k-j+1}'\big) \\ &\quad + c_{0,2j-1}\, p_{1,k-j+1}^2 + 2c_{0,2j-2}\, p_{1,k-j+1}\, p_{1,k-j+2} + c_{0,2j-3}\big(p_{1,k-j+2}^2 + 2p_{1,k-j+1}\, p_{1,k-j+3}\big) \\ &\quad\quad + \cdots + c_{0,1}\big(p_{1,k}^2 + 2p_{1,k-1}\, p_{1,k+1} + \cdots\big) + c_{0,0}\big(2p_{1,k}\, p_{1,k+1} + 2p_{1,k-1}\, p_{1,k+2} + \cdots\big) = 0 \end{aligned}$$

- Step 2, the coefficient of $z^{2k+2}u^2$:

$$\begin{aligned} -2\big(a_{1,k+1}p_{1,k-j+1} + a_{1,k}p_{1,k-j+2} + \cdots + a_{1,0}p_{1,2k-j+2}\big) + \big(p_{2,2k-j+2} - p'_{2,2k-j+2}\big) \\ + c_{0,2j}\, p^2_{1,k-j+1} + 2c_{0,2j-1}\, p_{1,k-j+1}\, p_{1,k-j+2} + c_{0,2j-2}\big(p^2_{1,k-j+2} + 2p_{1,k-j+1}\, p_{1,k-j+3}\big) \\ + \cdots + c_{0,0}\big(p^2_{1,k+1} + 2p_{1,k}\, p_{1,k+2} + \cdots\big) = 0 \end{aligned}$$

- Step s, the coefficient of $z^{2k+s}u^2$ for $1 \leq s \leq 2j - 2k - 1$, until…

- Step $2j - 2k - 1$, the coefficient of $z^{2j-1}u^2$:

$$\begin{aligned} -2\big(a_{1,2j-k-2}p_{1,k-j+1} + a_{1,2j-k-3}p_{1,k-j+2} + \cdots + a_{1,0}p_{1,j-1}\big) + \big(p_{2,j-1} - p'_{2,j-1}\big) \\ + c_{0,2j}\big(2p_{1,-1}\, p_{1,0} + \cdots\big) + c_{0,2j-1}\big(p^2_{1,0} + 2p_{1,-1}\, p_{1,1} + \cdots\big) + \cdots + c_{0,1}p^2_{1,j-1} = 0 \end{aligned}$$

Now *assume* $k \leq j - 1$, that is, assume that j is large. Then a term in $p^2_{1,0}$ appears with some $c_{0,s}$ in each of the above equations. Thus, choosing c_0 appropriately, we can solve them all, and we conclude that there are only restrictions on p'_2 when $p_{1,0} = 0$. Now, we can carry out a similar argument for the coefficients $p_{1,s}$ for each $0 \leq s \leq k$, so the set of non-generic bundles lives on the subvariety singled out by the equations

$$p_{1,0} = p_{1,1} = \cdots = p_{1,k} = 0\,,$$

thus having codimension at least $k + 1$.

In the remaining case where $k = j$, we see directly from (2.9) that $p(z,u) = \sum_{s=1}^{k-1} p_{1s}\, z^s\, u$. Thus the only non-generic bundle is the split bundle, and the generic set of $\mathfrak{M}(Z_k; k)$ is precisely $\mathbb{P}^{k-2}$ when $k \geq 2$, and empty when $k = 1$. (The same argument shows that the generic set is all of $\mathbb{P}^{2j-k-2}$ for $j \leq k \leq 2j - 2$.) □

We use this result to endow the set of generic bundles in $\mathfrak{M}(Z_k; j)$ with the structure of a smooth manifold and a quasi-projective variety.

It is possible to embed the moduli $\mathfrak{M}(Z_k; j)$ into $\mathfrak{M}(Z_k; j + k)$ as a topological space via two *elementary transformations* and a twist. An elementary transformation $\mathrm{Elm}_{\mathcal{L}}$ is a map parametrised by a torsion sheaf $\mathcal{L}$ whose restriction $\mathcal{L}|_D$ to a divisor D is locally free, and which modifies a vector bundle only over D – we explain this map in detail in the next subsection. Now let $\Phi\colon \mathfrak{M}(Z_k; j) \to \mathfrak{M}(Z_k; j+k)$ be defined by

$$\Phi(E) = \mathrm{Elm}_{\mathcal{O}_\ell(j+k)}\big(\mathrm{Elm}_{\mathcal{O}_\ell(j)}(E)\big) \otimes \mathcal{O}(-k)\,.$$

In coordinates, Φ sends the bundle given by (j, p) to $(j+k, z^k u^2 p)$. Using the additional structure provided by Theorem 2.25, we see that this map is in fact smooth and analytic.

Theorem 2.26. *The map Φ is well defined, injective and a homeomorphism onto its image, which consists of all bundles in $\mathfrak{M}(Z_k; j+k)$ that split on the second infinitesimal neighbourhood of ℓ.*

Proof. This is Theorem 4.12 in [BGK1] and follows essentially from earlier work by Gasparim. □

It is an open question how Φ affects the numerical invariants. Computational evidence suggests that if E is of splitting type j on Z_k, then

$$\chi(\ell, \Phi E) = \chi(\ell, E) + k + 2j\,,$$

although we do not have a proof at this point.

2.6.1 Elementary transformations

An important tool in the study of vector bundles and sheaves is the elementary transformation, which changes a locally free sheaf over a divisor. It works as follows. Let W be an algebraic variety, $D \subset W$ a Cartier divisor and $\mathcal{L} \in \mathrm{Pic}(D)$ a fixed line bundle on D. If $\mathcal{E}$ is any locally free sheaf on W and $r\colon \mathcal{E} \to \mathcal{L}$ a surjection that restricts to a surjection $\rho\colon \mathcal{E}|_D \to \mathcal{L}$, then $\mathcal{E}' := \mathcal{K}er(r)$ is called the *elementary transformation* of $\mathcal{E}$ induced by r, written $\mathrm{Elm}_{\mathcal{L}}(\mathcal{E})$. Since the divisor D is Cartier, $\mathcal{E}'$ is locally free. Writing $\mathcal{L}' := \mathcal{K}er(\rho)$, we obtain the *display* of the elementary transformation:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \mathcal{L}' \arrow[u] \arrow[r] & \mathcal{E}|_D \arrow[u] \arrow[r, "\rho"] & \mathcal{L} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{E}' \arrow[u, "t"] \arrow[r, "F"'] & \mathcal{E} \arrow[u] \arrow[r, "r"'] & \mathcal{L} \arrow[u, equal] \arrow[r] & 0 \\
 & \mathcal{E}(-D) \arrow[u] \arrow[r, equal] & \mathcal{E}(-D) \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Note that the induced surjection $t\colon \mathcal{E}' \to \mathcal{L}'$ gives the inverse elementary transformation (up to twisting by D).

In local coordinates. Our spaces Z_k have one compact divisor $\ell \cong \mathbb{P}^1$ given by

$$0 \longrightarrow \mathcal{O}_{Z_k}(-k) \longrightarrow \mathcal{O}_{Z_k} \longrightarrow \mathcal{O}_\ell \longrightarrow 0\,,$$

In our canonical local coordinates, ℓ is given by $\{u = 0\}$ on the chart U and by $\{z^k u = 0\}$ on the chart V, so the left map is just multiplication by u or $z^k u$ on the respective charts. Since every rank-2 bundle $\mathcal{E}$ on Z_k comes with a surjection $\mathcal{E} \to \mathcal{O}(j)$, restriction to ℓ gives a surjection

$$r\colon \mathcal{E}|_\ell \longrightarrow \mathcal{O}_{\mathbb{P}^1}(j)\,,$$

and we are in a position to apply an elementary transformation with respect to r to the bundle $\mathcal{E}$. In coordinates, r maps a local section (a, b) to the residue of b modulo (u) on the U-chart. The kernel of r (which is $\mathcal{E}'$) thus consists of all sections (a, b) for which b vanishes on ℓ. If $\mathcal{E}$ is given by (2.1), then $\mathcal{E}'$ is an extension

$$0 \longrightarrow \mathcal{O}(-j) \longrightarrow \mathcal{E}' \longrightarrow \mathcal{O}(j+k) \longrightarrow 0\,,$$

Thus $\mathcal{E}'$ has transition function

$$T' = \begin{pmatrix} z^j & p' \\ 0 & z^{-j-k} \end{pmatrix}\,,$$

and the inclusion $F = (f, \tilde{f})\colon \mathcal{E}' \to \mathcal{E}$ is given by $f(a, b) = (a, ub)$ on U and $\tilde{f}(A, B) = (A, z^k uB)$ on V. Since we must have $T \circ f = \tilde{f} \circ T'$, we compute

$$T \circ f \begin{pmatrix} a \\ b \end{pmatrix} = T \begin{pmatrix} a \\ ub \end{pmatrix} = \begin{pmatrix} z^j a + upb \\ z^{-j} ub \end{pmatrix} \qquad \text{and} \qquad \tilde{f} \circ T' \begin{pmatrix} a \\ b \end{pmatrix} = \tilde{f} \begin{pmatrix} z^j a + p'b \\ z^{-j-k} ub \end{pmatrix} = \begin{pmatrix} z^j a + p'b \\ z^{-j} ub \end{pmatrix}\,,$$

and thus $p' = up$.

The new bundle $\mathcal{E}'$ now comes with a surjection to $\mathcal{O}_{\mathbb{P}^1}(j+k)$, so we can perform another elementary transformation to arrive at a bundle $\mathcal{E}''$ with transition function

$$T'' = \begin{pmatrix} z^j & u^2 p \\ 0 & z^{-j-2k} \end{pmatrix}\,.$$

Finally, $\mathcal{E}''(-k)$ is a rank-2 bundle with vanishing first Chern class and splitting type $j+k$, and we see that the map $\mathcal{E} \mapsto \mathcal{E}''(-k)$ is given in coordinates by $p(z,u) \mapsto z^k u\, p(z,u)$.

For completeness, we record that the inverse transformation is given by the surjection $t\colon \mathcal{E}' \to \mathcal{L}' \cong \mathcal{O}_{\mathbb{P}^1}(-j)$. The map t is given on the U-chart by mapping (a,b) to the residue of a modulo (u), and on the V-chart by mapping $(\tilde{a},\tilde{b})$ to the residue of $\tilde{a}$ modulo $(z^k u)$.

2.6.2 Hausdorff strata

The main application of Theorem 2.26 is the stratification of $\mathfrak{M}(Z_k; j)$ into Hausdorff strata by fixing both the height and the width in the case where $j = nk$ for some n. This appeared in [BGK1, Section 4].

Theorem 2.27 (See [BGK1, Theorem 4.15]). *If $j = nk$ for some $n \in \mathbb{N}$, then the pair (h_k, w_k) stratifies the moduli spaces $\mathfrak{M}(Z_k; j)$ into Hausdorff components.*

2.7 Stability via the endomorphism bundle

Classical deformation theory of vector bundles on a (compact) surface X says that the obstruction to deforming a bundle $E \to X$ live in the second cohomology $H^2(X; \mathcal{E}nd\, E)$ (see [FK74], and the moduli space is smooth if this obstruction vanishes. In this case, the tangent space to the moduli space at E is the space of first-order deformations of E, $H^1(X; \mathcal{E}nd\, E)$, modulo the space of trivial deformations (i.e. deformations into isomorphic bundles) $H^0(X; \mathcal{E}nd\, E)$.

In our case we take $X = Z_k$, which is of cohomological dimension one, so the second cohomology of all coherent sheaves vanishes. Since Z_k is not compact, we cannot conclude that the moduli of vector bundles is a smooth space, and we already saw that this is not the case even for $SL(2)$-bundles. However, the bundle (or sheaf) $\mathcal{E}nd\, E$ still contains valuable numerical information, which in fact turns out to be equivalent to the information given by the width and height for instanton bundles. However, this perspective offers another interpretation of the invariants, and we may ask for a physical interpretation of the non-instanton bundles.

To be precise, we define two numbers that we will suggestively call h^1 and h^0. This notation is concise at the risk of being confusing, but the context should make clear what is meant. First off, since Z_k is the total space of a negative bundle over $\mathbb{P}^1$, the cohomology of $\mathcal{E}nd\, E$ vanishes in dimensions ≥ 2 and is finite-dimensional in dimension 1. Next we consider the zeroth cohomology of $\mathcal{E}nd\, E$. It is infinite-dimensional, since Z_k is non-compact and H^0 is the space of global sections. However, the *difference* of dimensions of H^0 for two different bundles is finite in a certain sense:

Consider the restriction of E to the m^{th} infinitesimal neighbourhood of ℓ. This space is projective and so

$$V_m(E) := H^0\big(\ell^{(m)}; \mathcal{E}nd(E)|_{\ell^{(m)}}\big)$$

is finite-dimensional, although the dimension of this space grows with m. But for each fixed m, we can compare the dimensions of $V_m(E)$ and V_m of the split bundle:

Definition 2.28. If E is a rank-2 bundle on Z_k with $c_1(E) = 0$, let $j \in \mathbb{Z}_{\geq 0}$ such that $E|_\ell \cong \mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$. We define $E_{\text{split}} := \mathcal{O}(-j) \oplus \mathcal{O}(j)$, that is, E_{split} is the split bundle of the same splitting type as E.

We compare the dimensions of $V_m(E)$ and $V_m(E_{\text{split}})$ as m increases. This difference of dimensions is independent of m for large values m. Thus we define:

Definition 2.29.

$$\begin{aligned} h^1(E) &:= h^1\big(Z_k; \mathcal{E}nd(E)\big)\,, \text{ and} \\ h^0(E) &:= h^0\big(\ell^{(m)}; \mathcal{E}nd(E_{\text{split}})|_{\ell^{(m)}}\big) - h^0\big(\ell^{(m)}; \mathcal{E}nd(E)|_{\ell^{(m)}}\big)\,, \end{aligned}$$

where m is taken sufficiently large so that the expression for $h^0(E)$ becomes constant eventually, which happens for $m \geq (4j-2)/k$.

Remark 2.30. The numbers $h^0(E)$ and $h^1(E)$ are analytic invariants of E, and they are in fact equivalent to $w_k(E), h_k(E)$ on instanton bundles, where $j = nk$ for some n, via the following relations:

$$w_k(E) + h_k(E) = \chi^{\text{loc}}(E) = \big((h^1(E) - h^0(E)) - j\big)/2 + j/k \tag{2.15}$$

$$h^0(E) + h^1(E) = h^1\big(\mathcal{E}nd(E_{\text{split}})\big) \tag{2.16}$$

This gives

$$h^0(E) = n^2k - \chi^{\text{loc}}(E)\,, \quad h^1(E) = kn(n+1) - 2n + \chi^{\text{loc}}(E)\,.$$

For non-instanton bundles, though, the numbers $h^0(E)$ and $h^1(E)$ (at least one of them, as they are not independent) provide information in addition to $w(E), h(E)$. Equation 2.16 follows from a simple Hilbert polynomial argument that we shall meet again in the proof of Proposition 3.16. On the other hand, Equation 2.15 has merely been shown by numeric computations, which is why we leave this statement as a remark without formal proof.

In view of the fact that for instanton bundles we can recover w_k, h_k from the single number h^1 and the fixed number $h^1 + h^0 = h^1\big(\mathcal{E}nd(\mathcal{O}(-j) \oplus \mathcal{O}(j))\big) = n(2nk + k - 2)$, we can replace the definition for (h, w)-stability for instanton bundles (with $j = nk$) by declaring a bundle (h, w)-stable if h^1 attains its minimum, which is $kn(n+2) - 2n - 1$.

Referring to the example computations in Table 2.1, we observe that for *non-instanton* bundles, h^1 is not sufficient to pick out the (h, w)-stable bundle, and the pair (h_k, w_k) is not sufficient to pick out the split bundle, so that we ought to consider all three numbers h_k, w_k, h^1 as independent invariants.

2.A Sample computation on a surface

We compute explicitly the width and height for a simple, non-trivial example, namely of the bundle E of splitting type $j = 3$ on the space $Z_2 = \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-2)\big)$ given by $p(z, u) = u$, so E has transition matrix

$$T = \begin{pmatrix} z^3 & u \\ 0 & z^{-3} \end{pmatrix} .$$

The space Z_2 has coordinate charts $U = \big\{(z, u)\big\}$ and $V = \big\{(z^{-1}, z^2 u)\big\}$. The contraction of ℓ is

$$\pi \colon Z_2 \to X_2 = \operatorname{Spec} R \,, \text{ where } R = \mathbb{C}[x_0, x_1, x_2]\big/(x_0 x_2 - x_1^2) \,.$$

Width. To compute $Q = (\pi_* E)^{\vee\vee}\big/\pi_* E$, we first compute sections of E over $\ell^{(n)}$ for all n. This amounts to computing the space of sections (a, b) of E as *formal* power series $a, b \in \mathbb{C}[\![z, u]\!]$, subject to the condition that a, b be holomorphic in $\{z, u\}$ and $z^{-3}b, z^3 a + ub$ be holomorphic in $\{z^{-1}, z^2 u\}$. This implies that b has the following form:

$$\begin{aligned} b(z,u) \;&=\; b_{00} + b_{01}z + b_{02}z^2 + b_{03}z^3 + \cancel{b_{04}z^4} + \cancel{\cdots} \\ &+\; b_{10}u + \cdots + b_{15}z^5 u + \cancel{b_{16}z^6 u} + \cancel{\cdots} \\ &+\; b_{20}u + \cdots + b_{27}z^7 u^2 + \cancel{b_{28}z^8 u^2} + \cancel{\cdots} \\ &+\; \cdots \end{aligned}$$

All terms $z^s u^r$ in b with $s - 3 > 2r$ have to vanish. Now we can compute an expression for a. Since z^{s+3} is never holomorphic in $\{z^{-1}, z^2 u\}$ for $s \geq 0$, there are no terms on $\ell^{(0)}$. We are left with the following:

$$\begin{aligned} a(z,u) &= \cancel{a_{00}} + \cancel{\cdots} \\ &+ a_{10}u + a_{11}zu + \cdots \\ &+ a_{20}u^2 + a_{21}zu^2 + a_{22}z^2u^2 + \cdots \\ &+ \cdots \end{aligned}$$

Now we consider the expression $z^3a + ub$ and pick out every coefficient that must vanish, i.e. where $\deg_z > 2\deg_u$. The first few such terms are:

$$z^3u\big(a_{10} + b_{03}\big) = 0 \qquad z^4u\ a_{11} = 0 \qquad z^5u\ a_{12} = 0 \qquad \ldots$$

$$z^5u^2\big(a_{22} + b_{15}\big) = 0 \qquad z^6u^2\ a_{23} = 0 \qquad \ldots$$

Finally we can write down generators:

$$\beta_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \beta_1 = \begin{pmatrix} 0 \\ z \end{pmatrix} \qquad \beta_2 = \begin{pmatrix} 0 \\ z^2 \end{pmatrix} \qquad \alpha = \begin{pmatrix} u \\ -z^3 \end{pmatrix} \qquad \cancel{\alpha' = \begin{pmatrix} u^2 \\ 0 \end{pmatrix}}$$

We could have written down many more generators, but *over the space* X_2, i.e. over the ring

$$R = \left\{x_0 = u, x_1 = zu, x_2 = z^2u\right\}/\left(x_0x_2 - x_1^2\right),$$

everything else can be expressed in terms of $\beta_0, \beta_1, \beta_2, \alpha$. (For example, $\alpha' = x_0\alpha + x_1\beta_2$.) It remains to find the relations among the generators, and we arrive at the complete description of the R-module

$$M = \left\langle \beta_0, \beta_1, \beta_2, \alpha \right\rangle_R / \left(x_1\beta_0 - x_0\beta_1,\ x_2\beta_0 - x_1\beta_1,\ x_1\beta_1 - x_0\beta_2,\ x_2\beta_1 - x_1\beta_2\right).$$

Application of the Theorem on Formal Functions tells us (in a highly non-trivial fashion) that $Q \cong \operatorname{coker} M \xrightarrow{\mathrm{ev}} M^{\vee\vee}$. Thus we must compute $M^\vee$ and thence $M^{\vee\vee}$. A moment's thought shows:

$$M^\vee = \operatorname{Hom}_R(M, R) = \left\{\beta^\vee, \alpha^\vee\right\},$$

where

$$\beta^\vee = \left\{\beta_i \mapsto x_i, \alpha \mapsto 0\right\} \quad \text{and} \quad \alpha^\vee = \left\{\beta_i \mapsto 0, \alpha \mapsto 1\right\}.$$

$M^\vee$ is already free, so $M^{\vee\vee}$ is free as well, given by

$$M^{\vee\vee} = \left\{\beta^{\vee\vee} = \{\beta^\vee \mapsto 1, \alpha^\vee \mapsto 0\}, \alpha^{\vee\vee} = \{\beta^\vee \mapsto 0, \alpha^\vee \mapsto 1\}\right\} .$$

The evaluation map ev: $M \to M^{\vee\vee}$ acts as follows:

$$\mathrm{ev}(\alpha) = \alpha^{\vee\vee} \qquad \mathrm{ev}(\beta_i) = x_i\beta^\vee$$

Thus the only element in $M^{\vee\vee}$, seen as a $\mathbb{C}$-vector space, that is *not* in the image of ev is the element $1.\beta^\vee$, so $\mathrm{coker}(\mathrm{ev}) = \langle \beta^\vee \rangle_{\mathbb{C}}$, which has dimension one, so $w(E) = 1$.

Height. The height of E is $h(E) := h^0\big(X_2; R^1\pi_* E\big)$. But $h^0 = \dim H^0$ is just the dimension of the stalk $(R^1\pi_*E)_0$. Now the dimension is the same for the stalk of the sheaf and the stalk of the completion of the sheaf, and the latter is computed by the Theorem on Formal Functions:

$$\dim\big(R^1\pi_*E\big)_0 = \dim\big(R^1\pi_*E\big)^\wedge_0 = \dim\Big(\varprojlim_n H^1\big(\ell^{(n)}; E|_{\ell^{(n)}}\big)\Big) . \tag{2.17}$$

Since the limit stabilises at finite n, the computation is actually easy and amounts to computing $H^1\big(\widehat{\ell}; E\big)$, which we will do now.

An element of H^1 is a section over $U \cap V$ modulo holomorphic sections over U or over V. We simply write down all such sections. In our case we have exactly two of them:

$$\begin{pmatrix} z^{-1} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z^{-2} \\ 0 \end{pmatrix}$$

Thus $h(E) = 2$.

2.B Algorithmic computation of the invariants

Part of this research was the development of a computer algorithm for the automatic computation of $w(E)$ and $h(E)$ of a bundle $E(j,p)$ on Z_k, whose input is (k,p,j).

What follows is a somewhat lengthy and technical description of the theoretical algorithm and its implementation in *Macaulay 2*. This section may be skipped without consequences. The implementation is available for download from my website `http://www.maths.ed.ac.uk/~s0571100/Instanton/` (or an appropriate future location), and the content of this section has been submitted for publication.

For the remainder of this section, we consider a fixed bundle $E(j,p)$ on Z_k.

Computing the width

We want to compute the dimension of the vector space $Q_0 := \big((\pi_* E)^{\vee\vee} / \pi_* E\big)_0$, which is the stalk at 0 of the skyscraper sheaf Q defined by the exact sequence

$$0 \longrightarrow \pi_* E \xrightarrow{\text{ev}} (\pi_* E)^{\vee\vee} \longrightarrow Q \longrightarrow 0\,,$$

where $\pi\colon Z_k \to X_k$ is the contraction map. By definition, the $\mathcal{O}_{X_k}$-module structure on $\pi_* E$ is determined by the lifting map

$$\widetilde{\pi}\colon \mathcal{O}_{X_k} \to \pi_* \mathcal{O}_{Z_k}\,, \tag{2.18}$$

which is an isomorphism away from 0 and whose stalk at 0 is, in U-coordinates, just $\widetilde{\pi}(w_i) = z^i u$, where

$$\mathcal{O}_{X_k,0} =: S = \mathbb{C}\big\{w_0, w_1, \ldots, w_k\big\} \big/ \big(w_i w_j - w_{i+1} w_{j-1}\big)\,,$$

where the ideal contains all the indices $i = 0, \ldots, k-2$ and $j = i+2, \ldots, k$. (The space X_k is the affine cone over the rational normal curve of degree k.)

Thus we are lead to compute the space of sections of E, first as $\mathbb{C}[z,u]$-module and then as an S-module. Since we are using the Theorem on Formal Functions for the computation, we will actually be computing the $\mathbb{C}[\![z,u]\!]$- and $S^\wedge$-module structures. However, on any Noetherian locally ringed space $\big(X, \mathcal{O}\big)$ the completion $\widehat{\mathcal{O}}$ is a flat $\mathcal{O}$-module, and $\mathcal{F}^\wedge_x = \mathcal{F}_x \otimes_{\mathcal{O}} \widehat{\mathcal{O}}$ for any $\mathcal{O}$-sheaf $\mathcal{F}$ and all $x \in X$.

The computation of $\pi_* E$ proceeds in three steps: First we apply the Theorem on Formal Functions (Theorem 1.6) to express $(\pi_* E)_0$ in terms of the cohomology of E, i.e.

$$\big(\pi_* E\big)^\wedge_0 = \varprojlim_n H^0\big(\ell^{(n)};\ E^{(n)}\big)\,.$$

In local coordinates, elements of $H^0\big(\ell^{(n)};\ E^{(n)}\big)$ are sections $\sigma\colon Z_k \to E$ which are of the form $\sigma|_U = \big(a(z,u), b(z,u)\big)$, where a and b are *a priori* power series in

$$\mathcal{O}_{\ell^{(n)}}(U) = \bigoplus_{i=0}^{n} u^i . \mathbb{C}\big\{z\big\}\,.$$

However, we require that the local section patch correctly onto the other chart, so that $T\sigma = \left(z^j a + pb, z^{-j} b\right)$ is a holomorphic section of $E^{(n)}(V)$, i.e. holomorphic in $\left(z^{-1}, z^k u\right)$. This shows that a and b are in fact polynomials, i.e. they contain only finitely many non-zero powers of z.

For the second step, we have to show that we can compute the module structure of $(\pi_* E))^\wedge_0$ from a finite amount of data (essentially by only going up to a finite infinitesimal neighbourhood, but see the subsection "Computation of M"). To be slightly more precise, we will *not* compute $H^0\left(\ell^{(n)};\ E^{(n)}\right)$, but instead we will identify finitely many elements in $H^0\left(\ell^{(n)};\ E^{(n)}\right)$ that generate $(\pi_* E))^\wedge_0$ as a $\mathbb{C}[\![z,u]\!]$-module. (The fact that we can do this depends crucially on the structure of the space Z_k and the fact that the conormal bundle of $\ell \subset Z_k$ is ample.)

Finally, once we have computed

$$M := H^0\left(\ell^{(N)};\ E^{(N)}\right)$$

(for some sufficiently large N) as a $\mathbb{C}[\![z,u]\!]$-module, the third and final step is to find the $S^\wedge$-module structure on M induced by the lifting map (2.18). Here we exploit the fact that u is not a zero-divisor in $\mathbb{C}[\![z,u]\!]$ and that every element in $\mathbb{C}[\![z,u]\!]$ can be expressed in terms of $w_i = z^i u$ after multiplication by a sufficiently high power of u.

Description of the algorithm

The example of § 2.A suggests a general algorithm: We must consider two *polynomials* a and b, use the condition that $z^j a(z,u) + p(z,u)b(z,u)$ and $z^{-j} b(z,u)$ be holomorphic in $\left(z^{-1}, z^k u\right)$ to obtain relations on the coefficients a_{rs} and b_{rs}, thence create the S-module M, and finally compute the dimension of the quotient $M^{\vee\vee}/M$.

The crucial consideration is that we only need consider *finitely many* terms in $a(z,u)$ and $b(z,u)$, and this will suffice to describe the module structure of M. In other words, we guarantee that we can choose *a priori* polynomials

$$a(z,u) = \sum_{r=0}^{A_1} \sum_{s=0}^{A_2} a_{rs} z^s u^r \quad \text{and} \quad b(z,u) = \sum_{r=0}^{B_1} \sum_{s=0}^{B_2} b_{rs} z^s u^r ,$$

in which we treat the coefficients a_{rs} and b_{rs} as indeterminates, which together with the finitely many relations among them generate the module M. The bounds A_1, A_2, B_1 and B_2 will only depend on k, j and p, and they will be determined at the start of the algorithm. This is described in the next subsection.

Following the computation of the relations among the coefficients, we require a small, technical routine to convert the $\mathbb{C}[z,u]$-module into an S-module. These technical algorithms are described at the end of the subsection "Implementation". Finally, for the computation of the quotient $M^{\vee\vee}/M$ we use the same computational method that was described in [GS05, Lemma 2.1 (iii)].

Computation of M

Intuitively, it is clear that to compute M one has to write down "enough" terms of a and b, calculate $f := z^j a + pb$ and set to zero all terms in f that are not holomorphic in z^{-1} and $z^k u$. This gives a set of relations among the coefficients a_{rs} and b_{rs}, which in turn determines a set of sections that generate M. In this section we give precise instructions on how to find the relations among coefficients and how to build from them a generating set of sections.

First let us fix some notation: To each coefficient a_{rs} and b_{rs}, let us associate, respectively, "elementary" sections

$$\sigma(a_{rs}) := \sigma^a_{rs} := \begin{pmatrix} z^s u^r \\ 0 \end{pmatrix} \qquad \text{and} \qquad \sigma(b_{rs}) := \sigma^b_{rs} := \begin{pmatrix} 0 \\ z^s u^r \end{pmatrix}.$$

Then the generator associated to a relation $R = a_{rs} + \sum_{il} R_{il} b_{il} = 0$, where R_{il} is non-zero for at least one (i,l), is $\sigma(R) := -\sigma^a_{rs} + \sum_{il} R_{il}\sigma^b_{il}$. We denote by $\mathcal{R}$ the set of all such relations, so we may consider $\mathcal{R}$ to be the "solution set" of the holomorphy condition $T\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)/\Gamma(E;V) = 0$. With this notation, M is generated as a $\mathbb{C}[z,u]$-module by the set $G_{\mathcal{R}} := \{\sigma(R) : R \in \mathcal{R}\}$, and as an S-module by $G'_{\mathcal{R}} := \{\pi_* \sigma(R) : R \in \mathcal{R}\}$.

There are two problems one faces when restricting oneself to a (finite) polynomial, which we turn into

Objectives for the algorithm.

1. One must find all generators of M, i.e. one must ensure that $G_{\mathcal{R}}$ generates M. For example, on Z_2 with $p = 0$ and $j = 4$, the a_{20}-term contributes a free generator $(u^2, 0)$, which one could miss by only considering the $r = 0$ and $r = 1$ infinitesimal neighbourhoods for a.

2. One must find all relations between $b_{r's'}$- and a_{rs}-terms. Some b_{rs}-terms may appear to be free when one does not consider enough a_{rs}-terms. For example, on Z_2 with $j = 5$ and $p = u^2$, the term $b_{05}z^5$ may erroneously seem to constitute the free generator $(0, z^5)$ if one

does not include the second infinitesimal neighbourhood and finds $a_{20} + b_{05} = 0$, so that the actual generator is $(-u^2, z^5)$.

There exists a precise bound on the number of infinitesimal neighbourhoods which one needs to consider. By including terms from a higher neighbourhoods into the polynomials a or b, one may see new relations involving terms from lower neighbourhoods appear, but at the same time this will add new generating terms for which one might in turn be tempted to find new relations in even higher neighbourhoods. However, we have *a priori* bounds on the terms in a and b that ensure that we compute the correct module structure on M.

3. It is acceptable for $\mathcal{R}$ to contain too many relations involving terms in a. This happens when there are not enough terms in b to match. In [GS05] this was called a "fake relation". However, if $R \in \mathcal{R}$ is such a fake relation, and if by considering higher terms we would find the corresponding "real" relation to be R', then we can ensure that $\sigma(R')$ is already contained in the module generated by $G_{\mathcal{R}}$.

   This will inevitably be the case when p contains several terms of different degree in u: In that case one cannot possibly find all correct relations among a finite set of terms. The key is to allow high terms of a to be set to zero "erroneously", rather than to miss a relation between a term $b_{r's'}$ and a term a_{rs}. (The latter would cause us to add a wrong generator, while the former only removes a potential generator – but we are careful to miss only multiples of earlier generators.)

The range of coefficients which one needs to consider depends on the extension p:

Definition 2.31. Let $p \in \mathbb{C}[z, z^{-1}, u]$. We define:

- $\min_u$:= the minimal degree of u occurring in p,
- $\max_u$:= the maximal degree of u occurring in p,
- $\min_z$:= the minimal degree of z occurring in p, and
- $\max_z$:= the maximal degree of z occurring in p.
- If $p \equiv 0$, then all the above values would be $-\infty$; however, for this case we define $\min_u := 0$, which will later save us from having to consider this case separately.

For a given bundle E on Z_k determined by (j, p), there are immediate bounds on the number of degrees of z that need to be considered for each fixed r:

Proposition 2.32. *For any degree r of u and independent of p, only the terms*

$$b_{r0}u^r + \cdots + b_{r,kr+j}u^r z^{kr+j}$$

occur in b.

Proof. We require that $z^{-j}b(z,u)$ be holomorphic in z^{-1} and $z^k u$. By multiplying $\sum_{s=0}^{\infty} b_{rs}z^s u^r$ by z^{-j} we see that the only terms that are holomorphic in z^{-1} and $z^k u$ are those claimed. □

Proposition 2.33. *For any $r < \min_u$, the terms in a of degree r in u, if any, are*

$$a_{r0}u^r + \cdots + a_{r,kr-j}u^r z^{kr-j} .$$

Proof. Since $r < \min_u$, no term $a_{rs}u^r z^s$ can be combined with any term in pb, so the problem reduces to making $z^j a_{rs}u^r z^s$ holomorphic in $z^k u$, which results precisely in those terms stated. □

Proposition 2.34. *For $r \geq \min_u$, the only terms $a_{rs}u^r z^s$ that can possibly be non-zero satisfy*

$$0 \leq s \leq \max\left\{k(r - \min_u) + j + \max\{\max_z, 0\},\ kr - j\right\} .$$

Proof. Consider all terms in $z^j a_{rs}u^r z^s$ that are not holomorphic in z^{-1} and $z^k u$: They must vanish unless they can be matched by a term in pb. The only terms in pb that have degree r in u are of the form $b_{r's'}u^{r'}z^{s'}$, where $r - \max_u \leq r' \leq r - \min_u$. Since the terms in b are as in Proposition 2.32, s has to run at least up to $kr'_{\max} + j = k(r - \min_u) + j$, but the multiplication pb may have shifted the term matching a_{rs} by up to $\max(0, \max_z)$ places up, which explains the first term in the statement.

Secondly, terms up to $s = kr - j$ are automatically holomorphic in the expression $z^j a$, so if $kr - j$ is greater than the previous expression, all terms up to $kr - j$ must be considered, and all the coefficients are free. □

Finally, we must turn the Objectives 1–3 into ranges for r that we choose to consider.

Proposition 2.35. *By considering only a truncated generic section*

$$\begin{pmatrix} a \\ b \end{pmatrix} = \sum_{r=0}^{\min_u - 1} \sum_{s=0}^{kr-j} a_{rs} \begin{pmatrix} z^s u^r \\ 0 \end{pmatrix} + \sum_{r=\min_u}^{\alpha} \sum_{s=0}^{\beta} a_{rs} \begin{pmatrix} z^s u^r \\ 0 \end{pmatrix} + \sum_{r=0}^{\gamma} \sum_{s=0}^{kr+j} b_{rs} \begin{pmatrix} 0 \\ z^s u^r \end{pmatrix} , \tag{2.19}$$

where

$$\begin{aligned}
\alpha &:= \max\left\{\lceil j/k \rceil, \max_u\right\} + \min_u\,, \\
\beta &:= \max\left\{kr - j,\ k(r - \min_u) + j + \max\{\max_z, 0\}\right\}\,, \textit{ and} \\
\gamma &:= \max\left\{\lceil j/k \rceil, \max_u\right\}\,,
\end{aligned}$$

one finds enough generators to compute the S-module M.

Remark 2.36. This statement contains two facts: First, we claim that our choice of polynomials a and b gives enough coefficients from which we form the generators $G_{\mathcal{R}}$ of M. Secondly, we claim that the set $\mathcal{R}$ of relations is *correct* in the following sense: If we set $A = B = \infty$ and if $\mathcal{R}_\infty$ denotes the associated set of relations, then one of two things happens for each $R \in \mathcal{R}_\infty$: Either R is already a relation in $\mathcal{R}$, or $\sigma(R)$ is an S-multiple of $\sigma(R')$ for some $R' \in \mathcal{R}$. (This case was illustrated in Objective 3.)

Proof. Let us denote the three big sums in (2.19) by Σ_1, Σ_2 and Σ_3 respectively from left to right. Since the section (a, b) is holomorphic on U, we must have $s \geq 0$, and the upper bounds for s in each of the three sums is given respectively by Propositions 2.33, 2.34 and 2.32. To justify the choice of the remaining bounds, consider the condition

$$T \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} z^j\, a + p(z,u)\, b \\ z^{-j} b \end{pmatrix} \in \Gamma(E; V)\,.$$

The term Σ_1 is seen to contribute free generators of M, since no term in $T\Sigma_1$ can be matched by any term from $T\Sigma_3$.

The important choice is that of the bound α. Once this has been chosen, we will only consider b up to u-degree $\alpha - \min_u$, such that $p(z,u)\, b$ is matched with a (this justifies Objective 3 above). This will justify the choice of $\gamma = \alpha - \min_u$. Moreover this ensures that there cannot be any generators coming from a that are erroneously considered as free. It remains to prove that our choice of the bound α leads to correct computation of the module M.

By construction, all the generators we get from a are correct, while the generators coming from b are either correct or fake. We have to show two things: (i) all fake relations are multiples of genuine relations, and (ii) any relation of M is a multiple of a relation that we have already found. But both (i) and (ii) follow directly from the choice of α. $\square$

Computation of $M^{\vee\vee}$ and $l(Q)$

This short subsection is merely included for completeness. It is no computational obstacle to compute the dual and bi-dual of M:

$$M^{\vee} := \mathrm{Hom}_S\big(M, S\big) \qquad \text{and} \qquad M^{\vee\vee} := \mathrm{Hom}_S\big(M^{\vee}, S\big)\,.$$

The *evaluation map* $\mathrm{ev}\colon M \hookrightarrow M^{\vee\vee}$ is the natural map given by

$$\mathrm{ev}(a)\colon \phi \mapsto \phi(a) \quad \text{for all } a \in M, \phi \in M^{\vee}.$$

Lastly, note that dimension is invariant under completion, i.e. $\dim Q_0^{\wedge} = \dim Q_0$, so we have $l(Q) = \dim\big(\mathrm{coker}(\mathrm{ev})\big)$.

Implementation of the algorithm

Input and output. The algorithm takes as input the data (k, p, j), where $k > 0$ and $j \geq 0$ are integers and p is a polynomial in $\big(z^{\pm 1}, u\big)$. The main function `iWidth(k,p,j)` computes the width $l(Q)$ for the bundle E on Z_k determined by (j, p).

Auxiliary functions. The main function `iWidth(k,p,j)` calls several auxiliary functions: The function `makeSectionsAndRing(k,p,j)` creates the polynomials $a(z,u)$ and $b(z,u)$ according to Propositions 2.32, 2.33 and 2.35. The function `getRelations(k,fTv)` computes the relations among the coefficients of a and b, where `fTv` $= z^j a(z,u) + p(z,u)b(z,u)$. (Note that `fTv` contains all the necessary information.) The function `makeModule` constructs the S-module M from the data `aPoly` and `bPoly`, which arise respectively from $a(z,u)$ and $b(z,u)$ by applying all the relations. (For example, if $a_{20} + b_{05} = 0$ is a relation, then we substitute $a_{20} \to b_{05}$ in a.) Finally, `qLength(M)` computes $l(Q)$ from the module M; for its implementation we refer to [GS05].

The main function. Name: `iWidth`. Input: `(k,p,j)`. Output: the instanton width $l(Q)$.
Pseudo code.

```
{aPoly, bPoly, allVars} := makeSectionsAndRing(k, p, j)
fTv    := z^j * aPoly + p * bPoly
relRes := getRelations(k, fTv)
```

apply substitutions from `relRes` to `aPoly` and `bPoly`

```
M       := makeModule(k, aPoly, bPoly, allVars)
return qLength(M)
```

Auxiliary function. Name: `makeSectionsAndRing`. Input: `(k,p,j)`. Output: the polynomials $a(z,u)$ and $b(z,u)$, and `allVars`, a collection of all coefficients occurring in $a(z,u)$ and $b(z,u)$.

Pseudo code.

minU := minimal u-degree of p

maxU := maximal u-degree of p

minZ := minimal z-degree of p

maxZ := maximal z-degree of p

```
aMax     := max(ceiling(j/k), maxU) + minU
bMax     := aMax - minU
if p = 0 then ( minU = 0; bMax = 0; aMax = ceiling(j/k))
```

generate coefficients:

a_{rs} such that $r = 0, \dots, \texttt{minU} - 1$ and $s = 0, \dots, kr - j$;

a_{rs} such that $r = \texttt{minU}, \dots, \texttt{aMax}$ and

$s = 0, \dots, \max\{kr - j, k(r - \texttt{minU}) + j + \max\{\texttt{maxZ}, 0\}\}$;

b_{rs} such that $r = 0, \dots, \texttt{bMax}$ and $s = 0, \dots, kr + j$.

```
aPoly    := sum(a_(r,s) z^s u^r)
bPoly    := sum(b_(r,s) z^s u^r)
```

allVars := collection of all coefficients `a_(r,s)` and `b_(r,s)`

```
return { aPoly, bPoly, allVars }
```

Main algorithm. Name: `getRelations`. Input: `(k,fTv)`, where `fTv` $= z^j a(z,u) + p(z,u)b(z,u)$. Output: A collection of relations like $\{a_{20} + b_{05},\ a_{31} + b_{14} + b_{06}\}$.

Synopsis. Each relation is the coefficient of a monomial $z^s u^r$ in `fTv` for which $s > kr$.

Pseudo code.

```
rels    := { }
```

```
expSet := set of exponents (r,s) appearing in fTv

for each (r,s) in expSet do
  if (s <= k * r) then continue
  term := the z^s u^r-term in fTv
  rel  := the coefficient of term, scaled to be monic
  rels := rels + { rel }
end for

return rels
```

Auxiliary function. Name: `makeModule`. Input: `(k, aPoly, bPoly, allVars)`. Here `aPoly` and `bPoly` are the results of substituting all the relations into the original $a(z,u)$ and $b(z,u)$. Output: the S-module M (e.g. its presentation matrix over S).

Synopsis. Iterating over each coefficient in `allVars`, we set this coefficient to 1 and all others to 0 to get a section (a,b) of E. By multiplying with a high power of u (called `uexp`), we can express $(u^N a, u^N b)$ as a section of $\pi_* E$, and those sections generate M.

Pseudo code.

```
S       := makeRing(k)
Smodule := image (0 0)^T: S^1 → S^2
N       := the maximum of s − kr over all monomial terms z^s u^r in aPoly and bPoly
uexp    := ceiling(N/k)
aPoly   := aPoly * u^uexp
bPoly   := bPoly * u^uexp

for each coefficient c in allVars do
  a := aPoly with c = 1 and all other coefficients = 0
  b := aPoly with c = 1 and all other coefficients = 0
  Smodule := Smodule + image (piStar(a,S) piStar(b,S))^T: S^1 → S^2
end do

return Smodule
```

This function calls two further auxiliary functions, `makeRing(k)` and `piStar`. The first one, `makeRing(k)`, returns the quotient ring $S := \mathbb{C}[w_0, \dots, w_k]/(w_i w_j - w_{i+1} w_{j-1})$ for $i = 0, \dots, k-2$ and $j = i+2, \dots, k$. The second function, `piStar`, converts monomials $u^r z^s$ into monomials $\prod_i w_i^{n_i}$ in S, where $\sum_i n_i = r$ and $\sum_i i n_i = s$. This is possible because we multiplied every term by the sufficiently high power $u^{\mathtt{uexp}}$.

Auxiliary function. Name: `piStar`. Input: `(p,S)`, where p is some polynomial in u and z in which each term is of sufficiently high degree in u, and `S` is the target ring. Output: The polynomial p expressed in w_i-coordinates, where $w_i = z^i u$.

Pseudo code.

```
res := 0                                  // this will store the result
k   := the number k if S is C[w_0,...,w_k]/(w_i w_j - w_{i+1} w_{j-1})
                                          // we have variables w_0, ..., w_k
for each term t in p do
  degU := u-degree of t
  degZ := z-degree of t
  fctr := 1

  if degZ > k * degU then
     error: this term is not convertible!
  end if

  diff := k
  while (diff != 0) do
    fctr := fctr * w_diff^(degZ/diff)   // degZ/diff is integer division
    degU := degU - (degZ/diff)
    degZ := degZ modulo diff
    diff := diff - 1
  end do

  fctr := fctr * w_0^degU
  res  := res + fctr * (coefficient of t)

  return res
```

At last, we need to compute the length of the module Q, which equals the dimension of $M^{\vee\vee}/M$ as a $\mathbb{C}$-vector space. The computation is performed by the function `qLength` using a presentation matrix for M; the actual algorithm is precisely the one described in [GS05, Lemma 2.1 (iii)].

Computing the height

The sheaf $R^1\pi_*E$ is supported at the origin, since π is an isomorphism everywhere else. Therefore $H^0\big(X_k; R^1\pi_*E\big) \cong \big(R^1\pi_*E\big)_0$. The Theorem of Formal Function gives

$$\big(R^1\pi_*E\big)^\wedge_0 = \varprojlim_n H^1\big(\ell^{(n)}; E^{(n)}\big)\,.$$

However, this limit stabilises at a finite n, and so we may simply compute the finite-dimensional vector space $H^1\big(Z_k; E\big)$; then its dimension is the height of E.

In this section we present an algorithm that produces a basis for $H^1\big(Z_k; E\big)$. For this we use the Čech description

$$H^1\big(Z_k; E\big) = \frac{\Gamma(E; U \cap V)}{\Gamma(E; U) \oplus \Gamma(E; V)}\,,$$

so we are looking for sections of E on the overlap $U \cap V$ modulo sections on either U or V. We recall that U and V are affine, and we may consider our sheaves either as analytic sheaves over complex spaces or as sheaves over algebraic schemes; both points of view give the same results.

Proposition 2.37 ([BGK1, Lemma 2.9]). *Let E be determined by (j, p). Then every* 1*-cocycle in $H^1\big(Z_k; E\big)$ can be represented locally over U as*

$$\sum_{r=0}^{\left\lfloor \frac{j-2}{k} \right\rfloor} \sum_{s=kr-j+1}^{-1} \begin{pmatrix} a_{rs} \\ 0 \end{pmatrix} z^s u^r\,. \tag{2.20}$$

The idea is the following: The vector space $H^1\big(Z_k; E\big)$ is certainly spanned by all the monomial cocycles $c_{rs} := (a_{rs} z^s u^r, 0)^T$ from Equation (2.20), so we need to identify which linear combinations of the c_{rs} vanish in cohomology. But c_{rs} vanishes in cohomology precisely if there is a function b holomorphic on U such that

$$T \begin{pmatrix} a_{rs} z^s u^r \\ b \end{pmatrix} = \begin{pmatrix} a_{rs} z^{s+j} u^r + pb \\ z^{-j} b \end{pmatrix} \tag{2.21}$$

is holomorphic on V. Since p is a polynomial, only finitely many terms in b need to be considered, and we obtain an algorithm.

First note that if $p = 0$, then there can be no relations among the c_{rs}, and $H^1\big(Z_k; E\big) = \langle\{c_{rs}\}\rangle_{\mathbb{C}}$.

Proposition 2.38. *If $p \neq 0$, let* $\min_u$ *be the smallest degree of u appearing in p, and write $\mu = \lfloor \frac{j-2}{k} \rfloor - \min_u$. To obtain $H^1\big(Z_k; E\big)$, it suffices to check Equation* (2.21) *for polynomials of the form*

$$b(z,u) = \sum_{r=0}^{\mu} \sum_{s=-j}^{kr} b_{rs} z^s u^r \ .$$

Proof. This is immediate from the form of p in (2.9) and Equation (2.21). □

Description of the algorithm

The algorithm itself consists of two parts: The first part computes all the linear relations between the generators c_{rs}; it returns a list of all basis elements for $H^1\big(Z_k; E\big)$ and a set of relations, which may contain lots of redundant information.

The second part of the algorithm takes these sets of generators and relations and reduces them to a minimal set of generators and relations. From this new data, we compute the dimension of $H^1\big(Z_k; E\big)$ as the minimal number of generators minus the minimal number of relations.

First part: finding relations

1. Let b be as in Proposition 2.38, treating all the coefficients as indeterminates.

2. For each monomial $z^s u^r$ for (r,s) in $\Big\{(r,s) : r = 0, \dots, \Big\lfloor \frac{j-2}{k} \Big\rfloor \text{ and } s = kr - j + 1, \dots, -1\Big\}$, do the following:

   (a) Let S be the set of all terms in pb with degree (r,s) in (u,z).

   (b) If $S = \varnothing$, then $c_{rs} := (a_{rs} z^s u^r, 0)^T$ is an independent generator of $H^1\big(Z_k; E\big)$.

   (c) Otherwise, if S is non-empty, let B be the set of all coefficients b_{il} appearing in S, and let $b' = \sum_{b_{il} \in B} b_{il} z^l u^i$. Note that $z^s u^r$ is proportional to at least one term of pb' by construction.

   (d) Remove from pb' all terms that are proportional to $z^s u^r$ and all terms which are holomorphic on U; call the result q.

(e) Finally, let Q be the set of terms in q that is *not* holomorphic on V. If $Q = \varnothing$, then the cycle c_{rs} vanishes in cohomology, otherwise we keep c_{rs} as a non-trivial generator and obtain the relation $z^s u^r + \sum_{t \in Q} t = 0$.

Implementation. Name: `iHeight`. Input: `(k,p,j)` corresponding to the bundle E determined by (j, p) on Z_k. Output: a pair `(G,R)`, where `G` is a set of monomials t such that the cycles $(t, 0)^T$ span a vector space V, and `R` is a set of linear relations on V (involving the coefficients b_{rs}) such that $H^1(Z_k; E) = V/R$.

Pseudo code.

```
minU     := minimal u-degree occurring in p
if p = 0 then minU = 0
bMax     := floor((j-2)/k) - minU
  make all indices b_(r,s) for r = 0,...,bMax and s = -j,...,k * r
bPoly    := ∑_{r,s} b_(r,s) * u^r * z^s
pb       := bPoly * p
aList    := list of terms u^r*z^s for r = 0,...,floor((j - 2)/k) and
s = k * r - j + 1,...,-1
aNonTrivials := {}              // These two variables
aRelations   := {}              // store the final result

for each aCycle in aList do
  // To begin, find all terms in p*b that cancel aCycle
  pbpruned := all terms from pb with the same (z,u)-degree as aCycle

  if (pbpruned = {}) then       // nothing can cancel aCycle
    aNonTrivials := aNonTrivials + {aCycle}
    continue
  else
    leftb   := all terms in bPoly that contain coefficients in pbpruned
    leftpb  := leftb * p
    aCycleR := (all terms in leftpb that are proportional to aCycle)
    leftpb  := leftpb - aCycleR
    leftpb  := leftpb - (all terms holomorphic in (z,u))
```

```
    leftovers := those terms of z^j * leftpb that are not holomorphic in (z^{-1}, z^k u)

    if leftovers = 0 then
      aNonTrivials := aNonTrivials + {aCycle}
      aRelations   := aRelations + {aCycleR + leftovers}
    end if

  end if

end for

return (aNonTrivials, aRelations)
```

Second part: reducing to minimal generators and relations The first part of the algorithm produces two sets of data: a set G of generating monomials of for form $z^s u^r$ (i.e. the cocycle $(z^s u^r, 0)^T$ is non-trivial in $H^1(Z_k; E)$, and a set R of relations which are polynomials in z, z^{-1}, u with coefficients b_{ij}. Let C be the set of all coefficients b_{ij} that can appear; C is determined by Proposition 2.38. To find minimal generators and relations, proceed as follows:

- Build a new set R'' of relations without indeterminates as follows: For each relation $r \in R$, for each $\beta \in C$, set $\beta = 1$ and all other coefficients in $C \setminus \{\beta\}$ to zero; add the relation $r|_{\beta=1, C\setminus\{\beta\}=0}$ to R''.
- Build a new set of generators G' and a new set of relations R' by starting with $G' = G$ and $R' = R''$ as follows: Let N be the set of monomial relations in R', i.e. relations of the form $z^s u^r = 0$. For each $r \in N$, remove r from G' and substitute $r = 0$ into every relation in R'. Let N be the new set of monomial relations in R' and repeat until $N = \varnothing$.
- The final set G' is a minimal set of generators, and the final set R' is a minimal set of relations.

Implementation in pseudo code. Name: `fixHeightRelations`. Input: `(G,R)`, the sets of generators and relations which the `iHeight` algorithm produced. Output: a new pair `(G',R')`, where `G'` is a minimal set of generators for the vector space $H^1(Z_k; E)$, and `R'` is a new set of linear relations, usually empty. Thus $|\mathtt{G}'| - |\mathtt{R}'|$ is the actual value of the height of E (this number is also returned in the actual implementation).

Pseudo code.

```
if (G = {} or R = {}) then
  return (G, R)
end if

rels     := {}      // this stores the result R
allvars := the set of coefficients b_(r,s)

for each term t in G do
  for each v in allvars do
    l1 := t  with v=1 and all other variables set to zero
    if l1 != 0 then rels = rels + {l1}
  end for
end for

prunednontrivs := G
prunedrels := rels
nullguys    := the set of one-term relations (e.g. z^3 u^2 = 0) in prunedrels

while (nullguys != {}) do
  for each term t in nullguys do
    replace (R + t) by (R) in prunedrels
    replace (R + t) by (R) in prunednontrivs
  end for
  nullguys := the set of one-term relations in prunedrels

end while

return (prunednontrivs, prunedrels)
```

Chapter 3

Threefolds

3.1 Introduction

In this chapter we study the local moduli problem on complex threefolds, and in the same way as in Chapter 2 we assume that our threefold W contains an embedded curve $\mathbb{P}^1 \cong \ell \subset W$, and the normal bundle of ℓ will play a crucial role. Unlike in the previous chapter, ℓ is not of middle dimension and there is no analogue of the self-intersection number, but instead we will consider whether ℓ moves in W or whether it is rigid, or even infinitesimally rigid. After establishing these properties, we continue to study the local moduli of bundles on W near ℓ as before.

An important ingredient in the study of the local moduli is extent to which a version of the GAGA principle holds on the spaces in question. Since the spaces are local $\mathbb{P}^1$s and GAGA holds on $\ell \cong \mathbb{P}^1$, we proceed by studying the infinitesimal neighbourhoods $\ell^{(m)}$ and the formal completion $\widehat{\ell}$. We consider three different examples: On the first space, bundles are filtered and algebraic, on the second they are filtered but not necessarily algebraic, and on the third there are rank-2 bundles that are not extensions.

In § 3.2 we define the spaces of interest and derive some explicit descriptions. We proceed to discuss the GAGA property of these spaces in § 3.3 before turning to the moduli problem proper. Endomorphism bundles are discussed in § 3.4 and numerical invariants in § 3.5 in preparation for the description of the moduli of bundles in the final § 3.6.

3.2 Local Calabi-Yau threefolds with rational curves

As we are studying spaces with an embedded compact curve $\mathbb{P}^1$, we reduce to the simplest such case, which is that of the total space of normal bundle of the line (also called a *local* $\mathbb{P}^1$ by algebraic geometers). The compact 2-cycles of such a space correspond to the holomorphic sections of the normal bundle, and they are all rationally equivalent to the zero section.

Local $\mathbb{P}^1$s are particularly amenable as they can be covered by two charts only; thus we perform explicit calculations. Moreover, all coherent cohomology groups vanish for all degrees but 0 and 1.

Motivated by the question how the moduli of bundles changes under birational transformation of the base, we are keeping in mind the question of whether ℓ may be contracted.

3.2.1 Definitions

We restrict our attention to local $\mathbb{P}^1$ spaces $W \cong N_{\ell/W}$ that are Calabi-Yau. Since

$$c_1(W) = c_1(\mathbb{P}^1) + c_1(N_{\ell/W}) \qquad \text{and} \qquad N_{\ell/W} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)\ ,$$

we have $a + b = -2$. When considering the contraction of a line inside a threefold, then only three essential local models may occur (e.g. see [Jim92]):

$$\begin{aligned} W_1 &:= \operatorname{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\big) \\ W_2 &:= \operatorname{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}\big) \\ W_3 &:= \operatorname{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)\big) \end{aligned}$$

In each case we denote by ℓ the zero-section, so that $\ell \cong \mathbb{P}^1$. The spaces have canonical charts $U \cong \mathbb{C}^2 \cong \{z, u, v\}$ and V, where respectively $V \cong \mathbb{C}^2 \cong \{z^{-1}, zu, zv\}$, $\{z^{-1}, z^2u, v\}$ and $\{z^{-1}, z^3u, z^{-1}v\}$. In each case, the canonical bundle is spanned globally by $dz \wedge du \wedge dv$, so we see explicitly that the spaces are Calabi-Yau. Note that the conormal sheaves of W_2 and W_3 are not ample. (More generally, all the spaces $W_i := \operatorname{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-i) \oplus \mathcal{O}_{\mathbb{P}^1}(i-2)\big)$ are Calabi-Yau, but we will not consider them, as they cannot appear as local models of the contraction of a smooth rational curve.)

3.2.2 Canonical forms

Let now E be a rank-2 bundle on one of the complex spaces W_i, $i = 1, 2, 3$ of splitting type j. Assume for now that E is an extension of line bundles

$$0 \longrightarrow \mathcal{O}(-j) \longrightarrow E \longrightarrow \mathcal{O}(j) \longrightarrow 0\ ,$$

where $\mathcal{O}(j)$ is just the pullback of $\mathcal{O}_{\mathbb{P}^1}(j)$, given by the transition matrix

$$T = \begin{pmatrix} z^j & p(z,u,v) \\ 0 & z^{-j} \end{pmatrix} .$$

It is necessary that p be of the form $p(z,u,v) = up'(z,u,v) + vp''(z,u,v)$, for otherwise the bundle E would in fact be of lower splitting type. This is an important point to which we return in the discussion of deformation spaces.

Proposition 3.1. *The extension class p can be reduced to the following form, respectively,*

$$\text{on } W_1: \qquad p(z,u,v) = \sum_{t=\epsilon}^{2j-2} \sum_{r=1-\epsilon}^{2j-2-t} \sum_{s=r+t-j+1}^{j-1} p_{trs}\, z^s u^r v^t ,$$

$$\text{on } W_2: \qquad p(z,u,v) = \sum_{t=\epsilon}^{\infty} \sum_{r=1-\epsilon}^{j-1} \sum_{s=2r-j+1}^{j-1} p_{trs}\, z^s u^r v^t , \text{ and}$$

$$\text{on } W_3: \qquad p(z,u,v) = \sum_{t=\epsilon}^{\infty} \sum_{r=1-\epsilon}^{\left\lfloor \frac{2j-2+t}{3} \right\rfloor} \sum_{s=3r-t-j+1}^{j-1} p_{trs}\, z^s u^r v^t ,$$

where $\epsilon \in \{0,1\}$.

Definition 3.2 (Canonical extension class). We call the form of p from Proposition 3.1 the *canonical form* of the extension.

Proof of Proposition 3.1. Suppose that E is an extension given by the transition matrix

$$T = \begin{pmatrix} z^j & p(z,u,v) \\ 0 & z^{-j} \end{pmatrix} .$$

A priori, p is given by a convergent power series

$$p(z,u,v) = \sum_{t=\epsilon}^{\infty} \sum_{r=1-\epsilon}^{\infty} \sum_{s=-\infty}^{\infty} p_{trs} z^s u^r v^t ,$$

where $\epsilon \in \{0,1\}$ accounts for the vanishing of p on the zero section $u = v = 0$. A bundle isomorphism casts T into the new form

$$T' = \begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z^j & p(z,u,v) \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{3.1}$$

where $\alpha, \beta, \gamma, \delta$ are holomorphic on V and A, B, C, D are holomorphic on U. In particular, we consider only $C = \gamma = 0$, whence $\alpha A = \delta D = 1$, and we can write

$$T' = \begin{pmatrix} \alpha & \beta \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} z^j & p(z,u,v) \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} \alpha^{-1} & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} z^j & p' \\ 0 & z^{-j} \end{pmatrix},$$

with

$$p' = \alpha B z^j + \beta D z^{-j} + \alpha D p\,. \tag{3.2}$$

We may fix $\alpha = D = 1$, say, and use β and B to remove terms from the power series of p:

First, any term $p_{trs} z^s u^r v^t$ with $s \geq j$ can be removed from p by setting $\beta = 0$ and $B = -p_{trs} z^{s-j} u^r v^t$; B is holomorphic on U. Thus we only need terms with $s \leq j-1$. Secondly, for fixed r and t, by setting $B = 0$ and $\beta = -p_{trs} z^{s+j} u^r v^t$ we remove terms with

$$\begin{aligned} r + t - j &\leq s \quad \text{on } W_1\,, \\ 2r - j &\leq s \quad \text{on } W_2\,, \text{ and} \\ 3r - t - j &\leq s \quad \text{on } W_3\,. \end{aligned}$$

Finally, we have obtained constraints for s, r and t for the remaining terms in p as follows:

$$\begin{aligned} r + t - j + 1 \leq s \leq j - 1 \quad &\Rightarrow \quad r + t \leq 2j - 2 \quad &&\text{on } W_1\,, \\ 2r - j + 1 \leq s \leq j - 1 \quad &\Rightarrow \quad r \leq j - 1 \quad &&\text{on } W_2\,, \text{ and} \\ 3r - t - j + 1 \leq s \leq j - 1 \quad &\Rightarrow \quad 3r - t \leq 2j - 2 \quad &&\text{on } W_3\,. \end{aligned}$$

The result follows immediately. □

Remark 3.3. If instead we also allow terms which are not multiples of u or v, we include extensions of lower splitting type. These more general functions are obtained by starting both sums over r and t at zero. We write, for example on W_1,

$$\widetilde{p}(z,u,v) = \sum_{t=0}^{2j-2} \sum_{r=0}^{2j-2-t} \sum_{s=r+t-j+1}^{j-1} p_{trs}\, z^s u^r v^t\,,$$

which determines an extension of splitting type $\leq j$ (but is not in canonical form if the splitting type is strictly less than j).

Corollary 3.4. *If $\lambda \neq 0$, then p and λp determine isomorphic bundles.*

Proof. In Equation (3.2), let $\beta = B = 0$, $\alpha = 1$ and $D = \lambda$. □

On the first infinitesimal neighbourhood $\ell^{(1)}$, the converse of Corollary 3.4 is true. We are working on the formal completion $\widehat{\ell}$, so local section of the structure sheaf $\mathcal{O}_{\widehat{\ell}}$ is a *formal* power series in u and v and *convergent* in z. Peternell's Existence Theorem (see Remark 1.7) asserts that a bundle on $\widehat{\ell}$ extends to a holomorphic bundle on an actual open (in the analytic topology) neighbourhood of ℓ. Thus we are allowed to let the entries of the matrices in the isomorphism (3.1) be formal power series. It is always possible to choose a nowhere vanishing formal power series with a finite number of prescribed coefficients, so that we can always make sure that the coordinate change matrices have nowhere vanishing determinant.

Proposition 3.5 (Bundles on $\ell^{(1)}$). *On the first infinitesimal neighbourhood $\ell^{(1)}$ in any of the three spaces W_i, $i = 1, 2, 3$, the only isomorphism of bundles is scaling. That is, two bundles $E_1|_{\ell^{(1)}}$ and $E_2|_{\ell^{(1)}}$ given by transition matrices*

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z^j & q \\ 0 & z^{-j} \end{pmatrix}$$

are isomorphic if and only if $q = \lambda p$ for some $\lambda \in \mathbb{C}^\times$.

Proof. The "if"-part is just Corollary 3.4.

For the "only if"-part (whose proof is similar to Proposition 2.20, and cf. [BGK1, Theorem 4.9]), first note that the restriction to the first neighbourhood $\ell^{(1)}$ implies that p and q only contain powers of u and v of total degree 1. It follows from Proposition 3.1 that p and q only contain certain powers of z^s, namely those with

$$\begin{array}{rll} & 2-j \le s \le j-1 & \text{on } W_1, \\ \text{in } z^s u: & 3-j \le s \le j-1 & \multirow{2}{*}{} \\ \text{in } z^s v: & 1-j \le s \le j-1 & \Big\} \text{ on } W_2, \text{ and} \\ \text{in } z^s u: & 4-j \le s \le j-1 & \\ \text{in } z^s v: & -j \le s \le j-1 & \Big\} \text{ on } W_3. \end{array} \tag{3.3}$$

Next, the two bundles are isomorphic only if there exist matrices holomorphic on the respective charts such that

$$\begin{pmatrix} z^j & p \\ 0 & z^{-j} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z^j & q \\ 0 & z^{-j} \end{pmatrix}.$$

Here A, B, C, D are power series on $U \cap \ell^{(1)}$ and $\alpha, \beta, \gamma, \delta$ on $V \cap \ell^{(1)}$, and we write, for instance, $A(z,u,v) = a_{00}(z) + a_{10}(z)u + a_{01}(z)v$ etc., where the coefficients are power series in z, and similarly for respectively $\alpha(z^{-1}, zu, zv)$, $\alpha(z^{-1}, z^2u, v)$ and $\alpha(z^{-1}, z^3u, z^{-1}v)$.

Comparing the two sides of the equation term by term gives four equations. We will only go through the case of W_1 here; for the other two just replace zu and zv in the following by z^2u and v for W_2 or by z^3u and $z^{-1}v$ for W_3.

$$\big(a_{00}(z) + a_{10}(z)u + a_{01}(z)v\big)z^j + pc_{00}(z) = \big(\alpha_{00}(z^{-1}) + \alpha_{10}(z^{-1})zu + \alpha_{01}(z^{-1})zv\big)z^j \tag{3.4}$$

$$z^{-j}\big(c_{00}(z) + c_{10}(z)u + c_{01}(z)v\big) = \big(\gamma_{00}(z^{-1}) + \gamma_{10}(z^{-1})zu + \gamma_{01}(z^{-1})zv\big)z^j \tag{3.5}$$

$$\begin{multline}\big(b_{00}(z) + b_{10}(z)u + b_{01}(z)v\big)z^j + pd_{00}(z) = \\ \alpha_{00}(z^{-1})q + \big(\beta_{00}(z^{-1}) + \beta_{10}(z^{-1})zu + \beta_{01}(z^{-1})zv\big)z^{-j} \tag{3.6}\end{multline}$$

$$\begin{multline}z^{-j}\big(d_{00}(z) + d_{10}(z)u + d_{01}(z)v\big) = \\ \gamma_{00}(z^{-1})q + \big(\delta_{00}(z^{-1}) + \delta_{10}(z^{-1})zu + \delta_{01}(z^{-1})zv\big)z^{-j} \tag{3.7}\end{multline}$$

The polynomials p and q are divisible by either u or v, so comparing the terms in (3.4) and (3.7) that are independent of both u and v gives $a_{00}(z) = \alpha_{00}(z^{-1})$ and $d_{00}(z) = \delta_{00}(z^{-1})$, whence all four are constants and $a_{00} = \alpha_{00}$ and $d_{00} = \delta_{00}$.

Next, equating terms in u or v in (3.6) gives

$$\big(b_{10}(z)u + b_{01}(z)v\big)z^j + pd_{00} = \alpha_{00}q + \big(\beta_{10}(z^{-1})zu + \beta_{01}(z^{-1})zv\big)z^{-j}\,.$$

By the conditions (3.3) on p and q, we must have

$$b_{10}(z)u + b_{01}(z)v = 0 \quad \text{and} \quad \beta_{10}(z^{-1})zu + \beta_{01}(z^{-1})zv = 0$$

(whence $b_{10} = 0 = b_{01}$ and $\beta_{10} = 0 = \beta_{01}$), and so $pd_{00} = \alpha_{00}q$.

The proof is finished by showing that $\alpha_{00}d_{00} \neq 0$. But the terms independent of both u and v in (3.6) yield $b_{00}(z)z^j = \beta_{00}(z^{-1})z^{-j}$, whence $b_{00} = 0 = \beta_{00}$. Thus over ℓ the coordinate change has determinant $a_{00}d_{00} = \alpha_{00}d_{00} \neq 0$. $\square$

Remark 3.6. Inspection of the proof of Proposition 3.1 shows that the conditions (3.3) and the equations (3.4)–(3.7) match up precisely, and that the conclusions of Proposition 3.5 are in fact valid for all spaces $W_i = \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-i) \oplus \mathcal{O}_{\mathbb{P}^1}(i-2)\big)$.

Remark 3.7. Another way of seeing extensions of the form (2.1) is by considering the isomorphism (see footnote on page 24

$$\mathrm{Ext}^1_{\mathcal{O}_{W_i}}\big(\mathcal{O}(j), \mathcal{O}(-j)\big) \cong H^1\big(W_i;\ \mathcal{O}(-j) \otimes \mathcal{O}(j)^\vee\big) \cong H^1\big(W_i;\ \mathcal{O}(-2j)\big)\,. \tag{3.8}$$

Direct computation shows that this is precisely the space of all coefficients in the generalised extension form from Remark 3.3, and the space of extensions E that satisfy $E|_\ell \cong \mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$ is thus precisely the subset of $\mathrm{Ext}^1\big(\mathcal{O}_{\mathbb{P}^1}(j), \mathcal{O}_{\mathbb{P}^1}(-j)\big)$ consisting of extension classes of the form $p(z,u,v) = up'(z,u,v) + vp''(z,u,v)$. Proposition 3.1 says that all terms in p that lie outside the given range are coboundaries with respect to this H^1.

In fact, computations of H^1-groups will be useful once more: The height of a rank-2 bundle E near an exceptional set, as defined by Equation (2.8), can be computed by the Theorem on Formal Functions (2.17) as follows:

$$\begin{aligned} h(E) &:= h^0\big(W'; R^1\pi_* E\big) = \dim H^0\big(W'; R^1\pi_* E\big) \\ &\qquad\qquad = \dim\big(R^1\pi_* E\big)_0 = \dim \left(\varprojlim_n H^1\big(\ell^{(n)}; E\big)\right)\,. \end{aligned}$$

But since E is algebraic in the cases which we consider (namely on W_1 and on the hypersurfaces D_i), the limit in the right-most term stabilises at finite n, and it remains to compute H^1 formally on $\widehat{\ell}$. To this end, we present a canonical form of 1-cocycles representing elements of $H^1\big(\widehat{\ell}; E\big)$:

Proposition 3.8 (Canonical cocycle). *Let E be an extension as in Proposition 3.1. A 1-cocycle $\sigma \in H^1(E)$ has the canonical representative, respectively,*

$$\text{on } W_1: \qquad \sigma = \sum_{t=0}^{j-2} \sum_{r=0}^{j-2-t} \sum_{s=r+t-j+1}^{-1} z^s u^r v^t \begin{pmatrix} a_{trs} \\ 0 \end{pmatrix},$$

$$\text{on } W_2: \qquad \sigma = \sum_{t=0}^{\infty} \sum_{r=0}^{\lfloor \frac{j-2}{2} \rfloor} \sum_{s=2r-j+1}^{-1} z^s u^r v^t \begin{pmatrix} a_{trs} \\ 0 \end{pmatrix}, \text{ and}$$

$$\text{on } W_3: \qquad \sigma = \sum_{t=0}^{\infty} \sum_{r=0}^{\lfloor \frac{t+j-2}{3} \rfloor} \sum_{s=3r-t-j+1}^{-1} z^s u^r v^t \begin{pmatrix} a_{trs} \\ 0 \end{pmatrix}.$$

Remark 3.9. Cocycles in $H^1\big(D_i; E\big)$ are obtained from this by setting $v = 0$.

Proof of 3.8. A priori, σ is given by

$$\sigma = \begin{pmatrix} a \\ b \end{pmatrix} = \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=-\infty}^{\infty} \begin{pmatrix} a_{trs} \\ b_{trs} \end{pmatrix} z^s u^r v^t .$$

Terms with non-negative powers of z are holomorphic on U, so we can restrict to $s \le -1$ and stay in the same cohomology class. Now on V,

$$T\sigma = \begin{pmatrix} z^j a + pb \\ \sum_t \sum_r \sum_{s=-\infty}^{-1} z^{s-j} u^r v^t \end{pmatrix} .$$

Since $j \ge 0$, the second entry is holomorphic on V, and $T\sigma$ is cohomologous to

$$T\sigma \sim \begin{pmatrix} z^j a + pb \\ 0 \end{pmatrix} .$$

Going back to U, we find

$$T^{-1} T\sigma \sim \begin{pmatrix} a + z^{-j} p \sum_t \sum_r \sum_{s=-\infty}^{-1} b_{trs} z^s u^r v^t \\ 0 \end{pmatrix} .$$

Since no power of z in p is greater than $j-1$, we can relabel the coefficients and write

$$T^{-1}T\sigma \sim \begin{pmatrix} \sum_{t=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=-\infty}^{-1} a'_{trs} z^s u^r v^t \\ 0 \end{pmatrix}.$$

Going to V one last time, we find that the terms in

$$z^j \sum_{t=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=-\infty}^{-1} a'_{trs} z^s u^r v^t$$

are holomorphic on V and can be discarded if

$$s+j \leq \begin{cases} r+t & \text{on } W_1, \\ 2r & \text{on } W_2, \text{ and} \\ 3r-t & \text{on } W_3. \end{cases}$$

This constrains the exponents as follows:

$$C-j+1 \leq s \leq -1 \text{ , where } C := r+t,\ 2r,\ 3r-t \text{ respectively.}$$

So $C \leq j-2$, and together with $r, t \geq 0$, the result follows. □

3.3 Algebraicity and filtrability

In Chapter 2 we made use of the fact that bundles on the surfaces Z_k were algebraically filtrable, which is a consequence of the ampleness of the conormal bundle of the compact line inside the total space. We can apply the exact same reasoning to derive similar results for the spaces W_1 and W_2 and to see why W_3 does not possess these properties.

Theorem 3.10. *Every holomorphic vector bundle on* $\mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\big)$ *is filtrable and algebraic.*

Proof. This is a direct application of Theorem 2.2. □

Theorem 3.11. *Let ℓ be the zero section of $\mathcal{O}_{\mathbb{P}^1}(-2)\oplus\mathcal{O}_{\mathbb{P}^1}(0)$. Fix an integer $r \geq 1$ and a holomorphic rank-r vector bundle E on $\widehat{\ell}$. Let $a_1 \geq \cdots \geq a_r$ be the splitting type of $E|_\ell$. Then there exist vector bundles F_i on $\widehat{\ell}$, $0 \leq i \leq r$, such that $F_r := E$, $F_1 := L_{a_1}$, $F_0 := \{0\}$ and $F_i|_\ell$ has rank i and splitting* *type $a_1 \geq \cdots \geq a_i$, and such that there are $r-1$ exact sequences on $\widehat{\ell}$ (for $2 \leq i \leq r$)*

$$0 \longrightarrow L_{a_i} \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow 0\,, \tag{3.9}$$

where $L_{a_i} \cong \mathcal{O}(a_i)$.

Proof. The result is obvious if $r = 1$. Hence we may assume $r \geq 2$ and that the result is true for all vector bundles with rank at most $r-1$. By assumption there is an injective map $j\colon \mathcal{O}_\ell(a_r) \to E|_\ell$ on ℓ such that $\operatorname{coker}(j)$ is a rank-$(r-1)$ vector bundle on ℓ with splitting type $a_1 \geq \cdots \geq a_{r-1}$. The map j gives a nowhere-zero section s of $E(-a_r)|_\ell$. Let us show that this section extends over a neighbourhood of ℓ: There is an exact sequence

$$0 \longrightarrow S^t(N^*_{\ell,W_2}) \longrightarrow \mathcal{O}_\ell^{(t+1)} \longrightarrow \mathcal{O}_\ell^{(t)} \longrightarrow 0\,, \tag{3.10}$$

where $S^t(N^*_{\ell,W_2})$ is the t^{th} symmetric power of the conormal sheaf of ℓ in W_2. In this case, we have $N_{\ell,W_2} \cong \mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell$, therefore,

$$S^t\big(N^*_{\ell,W_2}\big) \cong \bigoplus_{k=0}^{t} \mathcal{O}_\ell(2k)\,.$$

After tensoring by the bundle $E(-a_r)$, the exact sequence (3.10) becomes

$$0 \longrightarrow E(-a_r) \otimes \left(\bigoplus_{k=0}^{t} \mathcal{O}_\ell(2k)\right) \longrightarrow E(-a_r) \otimes \mathcal{O}_\ell^{(t+1)} \longrightarrow E(-a_r) \otimes \mathcal{O}_\ell^{(t)} \longrightarrow 0\,,$$

thus inducing the long exact cohomology sequence

$$\cdots \to H^0\big(\ell; E(-a_r) \otimes \mathcal{O}_\ell^{(t+1)}\big) \to H^0\big(\ell; E(-a_r) \otimes \mathcal{O}_\ell^{(t)}\big) \to \bigoplus_{k=0}^{t} H^1\big(\ell; E(-a_r + 2k)\big) \to \cdots.$$

Note that $H^0\big(\ell; E(-a_r) \otimes \mathcal{O}_\ell^{(t)}\big)$ is the space of global sections of $E(-a_r)$ on the t^{th} infinitesimal neighbourhood of ℓ in W_2; moreover, the obstruction to extending a section from the t^{th} infinitesimal neighbourhood to the $(t+1)^{\text{st}}$ one lives in

$$\bigoplus_{k=0}^{t} H^1\big(\ell; E(-a_r + 2k)\big)\,.$$

However, since $E(-a_r)$ is a bundle of degree $\sum_{i=1}^{r-1}(a_i - a_r) \geq 0$, $E(-a_r + 2k)$ is of non-negative degree for $0 \leq k \leq t$, and thus all the cohomology groups $H^1\big(\ell; E(-a_r + 2k)\big)$ vanish for $0 \leq k \leq t$. Thus any section of $E(-a_r)$ on the t^{th} infinitesimal neighbourhood extends to the $(t+1)^{\text{st}}$.

Hence, by Grothendieck's existence theorem ([Gr61, 5.1.4]), the section s extends to an actual neighbourhood of ℓ in W_2, and consequently there is an exact sequence on W_2 of the form

$$0 \longrightarrow L_{a_r} \longrightarrow E \longrightarrow F_{r-1} \longrightarrow 0\,.$$

□

Corollary 3.12. *Every algebraic vector bundle on* $W_2 = \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}\big)$ *is filtrable.*

Proof. Every such bundle is already determined on a finite infinitesimal neighbourhood of ℓ. □

Unlike in the case of W_1, however, there are non-algebraic bundles on $\widehat{\ell}$. It follows from Peternell's Existence theorem, though, that all such bundles do in fact extend to an analytic neighbourhood of ℓ.

Remark 3.13. The proof of the preceding theorem shows what goes wrong on W_3. There, since the conormal sheaf $N^*_{\ell,W_3} = \mathcal{O}_\ell(3) \oplus \mathcal{O}_\ell(-1)$ has a negative factor, the cohomology $H^1\big(\ell; E(-a_r) \otimes S^t(N^*_{\ell,W_3})\big)$ never vanishes, and so extending sections from $E_\ell(-a_r)$ to higher infinitesimal neighbourhoods of ℓ is generally always obstructed.

3.4 The endomorphism bundle

If the bundle E is given by transition matrix $T = \left(\begin{smallmatrix} z^j & p \\ 0 & z^{-j} \end{smallmatrix}\right)$, then the bundle $\mathrm{End}(E) = E \otimes E^*$ is given by the transition matrix $T \otimes T^T$. After a convenient change of coordinates given by

$$P := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = P^{-1}\,,$$

we will express the transition matrix of $E \otimes E^*$ as

$$S := P\big(T \otimes T^T\big)P = \begin{pmatrix} 1 & z^j p & z^{-j} p & p^2 \\ 0 & z^{2j} & 0 & z^j p \\ 0 & 0 & z^{-2j} & z^{-j} p \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{, so} \quad S^{-1} = \begin{pmatrix} 1 & -z^{-j} p & -z^j p & p^2 \\ 0 & z^{-2j} & 0 & -z^{-j} p \\ 0 & 0 & z^{2j} & -z^j p \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We are interested in $H^i\big(X;\mathcal{E}nd\,E\big)$ for $i = 0,1$ and $X = Z_k, W_1, W_2, W_3$. Like before, H^0 is the space of sections $(a,b,c,d) \in \Gamma(U;\mathcal{E}nd\,E) =: \Gamma_U$ such that $S(a,b,c,d) \in \Gamma_V$, while H^1 is the space of sections $\Gamma_{U\cap V}$ modulo $\Gamma_U \oplus \Gamma_V$.

A typical component of a section on U is given, say in the case where $X = Z_k$, by $a(z,u) = \sum_{r,s\geq 0} a_{rs} z^s u^r$, and a section on $U \cap V$ is given by $a(z,u) = \sum_{r\geq 0}\sum_{s=-\infty}^{\infty} a_{rs} z^s u^r$. We have

$$S\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a + z^j pb + z^{-j}pc + p^2 d \\ z^{2j} b + z^j pd \\ z^{-2j} c + z^{-j} pd \\ d \end{pmatrix}.$$

We use this explicit form in the sequel to compute numerical invariants of the endomorphism bundle $\mathcal{E}nd(E)$.

3.5 Numerical invariants

By contrast to the case of bundles on surfaces, it is rather more involved to find numerical invariants of bundles on our threefolds W_i, $i = 1,2,3$. First, only on W_1 is the zero section ℓ contractible, so the concepts of width and height only make sense on W_1, but not on W_2 or W_3. Moreover, for codimensional reasons the width always vanishes on W_1 (see [BGK2, Lemma 5.2]). In this section we define several new numerical invariants that contain geometric information and provide a way of partitioning the moduli. These new numbers are "partial" invariants arising from restricting to a subspace, and invariants of the endomorphism bundle.

The spaces W_i contain two distinguished prime Cartier divisors D_1 and D_2, which are given in our canonical coordinates by the equations $D_1 \cap U = \{v = 0\}$ and $D_2 \cap U = \{u = 0\}$ on the U-chart and by $D_1 \cap V = \{z^i v = 0\}$ and $D_2 = \{z^{2-i} u = 0\}$ on the V-chart. By restricting to these surfaces, we define the following *partial invariants*:

Definition 3.14.

$$\begin{aligned} w'(E) &= w(E|_{D_1}) \qquad & w''(E) &= w(E|_{D_2}) \\ h'(E) &= h(E|_{D_1}) \qquad & h''(E) &= h(E|_{D_2}) \end{aligned} \tag{3.11}$$

Note that on W_1 both divisors D_1 and D_2 are isomorphic to $Z_1 = \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1)\big)$. On W_2, we have $D_2 \cong Z_2 = \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-2)\big)$, but $D_1 \cong \mathbb{C}^2$, so we will only consider the restriction of bundles to D_2. (In fact, we have an entire families of divisors from the pencils spanned by D_1 and D_2. We will see this again later when we look at examples of moduli.)

Next we examine the endomorphism bundle $\mathcal{E}nd(E)$, which behaves very differently on W_1 and W_2. On W_1, the first cohomology group of $\mathcal{E}nd(E)$ is finite-dimensional, so its dimension is an invariant. We define:

$$\begin{aligned} h^1(E) &:= h^1\big(W_1; \mathcal{E}nd\, E\big) \\ \Delta_1 &:= h^1\big(W_1; \mathcal{E}nd(E_{\text{split}})\big) - h^1\big(W_1; \mathcal{E}nd(E)\big) \end{aligned} \tag{3.12}$$

The zeroth cohomology group of $\mathcal{E}nd\, E$ is infinite-dimensional, and we employ the same strategy as on Z_k: The infinitesimal neighbourhoods $\ell^{(m)}$ are projective schemes, so the restrictions of $\mathcal{E}nd(E)$ to them have finite-dimensional cohomologies. Also, we can compute those dimensions for the endomorphism bundle of the split bundle of the same splitting type as E and compare them. Let us fix notation for these split bundles:

Definition 3.15. If E is any bundle of splitting type j, we write $E_{\text{split}} := \mathcal{O}(-j) \oplus \mathcal{O}(j)$ for the split bundle of the same splitting type as E.

As m increases, the *difference* between the dimensions of the cohomologies is eventually constant, and this gives our second invariant:

$$\Delta_0 \ := \ \lim_{n\to\infty}\Big(h^0\big(W_1; \mathcal{E}nd(E_{\text{split}})|_{\ell^{(m)}}\big) - h^0\big(W_1;\ \mathcal{E}nd(E)|_{\ell^{(m)}}\big)\Big) \tag{3.13}$$

The numbers we have just defined are not independent and satisfy several relations.

Proposition 3.16. *For all rank-2 bundles E on $X = Z_k, W_1, W_2$,*

$$h^1\big(X; \mathcal{E}nd\, E|_{\ell^{(m)}}\big) - h^0\big(X; \mathcal{E}nd\, E|_{\ell^{(m)}}\big) = h^1\big(X; \mathcal{E}nd(E_{\text{split}})|_{\ell^{(m)}}\big) - h^0\big(X; \mathcal{E}nd(E_{\text{split}})|_{\ell^{(m)}}\big)\,.$$

Proof. We can express the statement in terms of the Hilbert polynomial

$$\phi_{\mathcal{F}^{(m)}}(n) := \chi\big(\mathcal{F}^{(m)}(n)\big) = h^1\big(X; \mathcal{F}|_{\ell^{(m)}}(n)\big) - h^0\big(X; \mathcal{F}|_{\ell^{(m)}}(n)\big)$$

for any coherent sheaf $\mathcal{F}$ on X; then the statement is

$$\phi_{\mathcal{E}nd\, E^{(m)}}(0) = \phi_{\mathcal{E}nd\, E^{(m)}_{\text{split}}}(0)\,.$$

But in fact we have $\phi_{\mathcal{E}nd\,E^{(m)}}(n) = \phi_{\mathcal{E}nd\,E^{(m)}_{\mathrm{split}}}(n)$ for any n and m, since the Hilbert polynomial is additive for short exact sequences of coherent sheaves over projective schemes, and $E^{(m)}$ is the extension

$$0 \longrightarrow \mathcal{O}_{\ell^{(m)}}(-j) \longrightarrow E^{(m)} \longrightarrow \mathcal{O}_{\ell^{(m)}}(j) \longrightarrow 0\,.$$

Thus both $\mathcal{E}nd\,E$ and $\mathcal{E}nd\,E_{\mathrm{split}}$ have a filtration

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow F_4 \longrightarrow 0\,,$$

where $F_1 \cong \mathcal{O}(-2j)$, $F_2/F_1 \cong \mathcal{O}$, $F_3/F_2 \cong \mathcal{O}$, $F_4/F_3 \cong \mathcal{O}(2j)$, and thus their Hilbert polynomials coincide. □

Corollary 3.17. *For all rank-2 bundles E on $X = Z_k, W_1$, we have $\Delta_0(E) = \Delta_1(E)$, or equivalently $\Delta_0(E) + h^1(E) = h^1\big(X; \mathcal{E}nd(E_{\mathrm{split}})\big)$.*

Proof. This follows from Proposition 3.16 by fact that $H^1(X; \mathcal{E}nd\,E)$ is already determined on a finite neighbourhood $\ell^{(m)} \subset X$ and by unravelling the definitions of Δ_0 and Δ_1. □

Proposition 3.18. *Suppose $(X, \mathcal{O}_X)$ is a projective scheme with a fixed, ample twisting sheaf $\mathcal{O}_X(1)$, and $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ a short exact sequence of coherent $\mathcal{O}_X$-sheaves. Then the Hilbert polynomials satisfy $\phi_{\mathcal{F}} = \phi_{\mathcal{F}'} + \phi_{\mathcal{F}''}$.*

Proof. By definition of ampleness, there exists a number n such that $\mathcal{E}(n)$ is generated by global sections for $\mathcal{E} = \mathcal{F}', \mathcal{F}, \mathcal{F}''$, and thus $H^i(X; \mathcal{E}) = 0$ for $i > 0$, and thus $\phi_{\mathcal{E}}(m) = h^0\big(X; \mathcal{E}(m)\big)$ for $m \geq n$. The short exact sequence of the hypothesis induces a long exact sequence

$$0 \longrightarrow H^0\big(X; \mathcal{F}'(m)\big) \longrightarrow H^0\big(X; \mathcal{F}(m)\big) \longrightarrow H^0\big(X; \mathcal{F}''(m)\big) \longrightarrow \cancel{H^1\big(X; \mathcal{F}'(m)\big)}\,,$$

and the result follows. □

Lemma 3.19. *Let E be an extension of type 2.1 with splitting type j on either Z_k or W_1. Then the Hilbert polynomial of $E|_{\ell^m}$ is*

$$n \mapsto \chi\big(E^{(m)}(n)\big) := \sum_i (-1)^i h^i\big(\ell^{(m)}; E(n)\big) = \begin{cases} (m+1)(km+2+2n) & \text{on } Z_k \text{, and} \\ \frac{1}{3}(m+2)(m+1)(2m+3n+3) & \text{on } W_1. \end{cases}$$

In particular, $\chi\big(E^{(m)}(n)\big)$ is independent of the extension class, and independent of the splitting type j. Similarly, the Hilbert polynomial of the endomorphism bundle $\mathcal{E}nd\,E|_{\ell^{(m)}}$ is $2\chi\big(E^{(m)}(n)\big)$.

Proof. It follows from the proof of Proposition 3.16 that the Hilbert polynomials in question are determined by the Hilbert polynomial of the line bundles $\mathcal{O}_{\ell^{(m)}}(p)$ for all p. Since $\mathcal{O}_{\ell^{(m)}}(1)$ is ample, the higher cohomology of $\mathcal{O}_{\ell^{(m)}}(p)$ vanishes for sufficiently large p. (We can verify this by direct computation.)

Being a polynomial, the Hilbert polynomial is determined by finitely many values, so it suffices to compute $\phi_{\mathcal{E}nd\,E^{(m)}}(n) = h^0\big(\ell^{(m)}; \mathcal{O}_{\ell^{(m)}}(n)\big)$ for large n. By the additivity of the Hilbert polynomial and the fact that E and $\mathcal{E}nd\,E$ have filtrations by line bundles (as given in the proof of Proposition 3.16) which restrict to filtrations on every infinitesimal neighbourhood $\ell^{(m)}$, we compute:

$$\begin{aligned}
\phi_{E^{(m)}}(n) &= \phi_{\mathcal{O}_{\ell^{(m)}}(-j)}(n) + \phi_{\mathcal{O}_{\ell^{(m)}}(j)}(n) \\
\phi_{\mathcal{E}nd\,E^{(m)}}(n) &= \phi_{\mathcal{O}_{\ell^{(m)}}(-2j)}(n) + 2\phi_{\mathcal{O}_{\ell^{(m)}}}(n) + \phi_{\mathcal{O}_{\ell^{(m)}}(2j)}(n)
\end{aligned}$$

We conclude this proof by computing $H^0\big(\ell^{(m)}; \mathcal{O}(p)\big)$. Now we have to consider the spaces Z_k and W_1 separately.

On $\ell^{(m)} \subset Z_k$, a section $a \in \mathcal{O}(p)(U)$ is $a(z,u) = \sum_{r=0}^{m} \sum_{s=0}^{\infty} a_{rs} z^s u^r$ such that $\sum_{r,s} a_{rs} z^{s-p} u^r$ is holomorphic in $(z^{-1}, z^k u)$, i.e. $s - p \leq kr$. Thus

$$a(z,u) = \sum_{r=0}^{m} \sum_{s=0}^{kr+p} a_{rs} z^s u^r \,,$$

which has $\frac{1}{2}(m+1)(km+2+2p) =: \phi_{\mathcal{O}}(p)$ coefficients.

On $\ell^{(m)} \subset W_1$, a section $a \in \mathcal{O}(p)(U)$ is $a(z,u,v) = \sum_{t=0}^{m} \sum_{r=0}^{m-t} \sum_{s=0}^{\infty} a_{trs} z^s u^r v^t$ such that $\sum_{t,r,s} a_{trs} z^{s-p} u^r v^t$ is holomorphic in (z^{-1}, zu, zv), i.e. $s - p \leq r + t$. Thus

$$a(z,u,v) = \sum_{t=0}^{m} \sum_{r=0}^{m-t} \sum_{s=0}^{r+t+p} a_{trs} z^s u^r v^t \,,$$

which has $\frac{1}{6}(m+2)(m+1)(2m+3p+3) =: \phi_{\mathcal{O}}(p)$ coefficients.

Putting it all together, we have

$$\begin{aligned}
\phi_{E^{(m)}}(n) &= \phi_{\mathcal{O}}(-j+n) + \phi_{\mathcal{O}}(j+n)\,, \\
\phi_{\mathcal{E}nd\,E^{(m)}}(n) &= \phi_{\mathcal{O}}(-2j+n) + 2\phi_{\mathcal{O}}(n) + \phi_{\mathcal{O}}(2j+n)\,,
\end{aligned}$$

which gives the desired functions. □

j	p	Δ_0	Δ_1	h^1	(w', h')	(w'', h'')	height
3	0	0	0	35	$(6,3)$	$(6,3)$	4
3	u	15	15	20	$(1,2)$	$(6,3)$	3
3	zu	15	15	20	$(1,2)$	$(6,3)$	3
3	$v+u$	15	15	20	$(1,2)$	$(1,2)$	3
3	$v+zu$	18	18	17	$(1,2)$	$(1,2)$	2
3	z^2u	10	10	25	$(3,2)$	$(6,3)$	3
3	$z^{-1}u$	10	10	25	$(3,2)$	$(6,3)$	3
3	$z^{-1}u+u$	15	15	20	$(1,2)$	$(6,3)$	3
3	$z^{-1}u+zu$	15	15	20	$(1,2)$	$(6,3)$	3
3	$z^{-1}u+z^2u$	15	15	20	$(1,2)$	$(6,3)$	3
3	$z^{-1}u+z^2v$	16	16	19	$(3,2)$	$(3,2)$	2
3	$z^{-1}v+z^2u$	16	16	19	$(3,2)$	$(3,2)$	2
3	$z^{-1}u+z^{-1}v$	10	10	25	$(3,2)$	$(3,2)$	3

Table 3.1: Example data on W_1. Observe that h^1 (or Δ_1) is a finer measure of genericity than the height.

On W_2 the situation is different. Since $W_2 \cong Z_2 \times \mathbb{C}$, the Künneth formula shows that both the zeroth and the first cohomology groups of $\mathcal{E}nd(E)$ are infinite-dimensional. But when we use the same strategy and compare the dimensions of the cohomology groups of the restrictions to the m^{th} infinitesimal neighbourhood of $\mathcal{E}nd(E)$ and $\mathcal{E}nd(E_{\text{split}})$, we find that their difference increases linearly in m.

In fact, more is true. Rearranging the equation of Proposition 3.16, we see that

$$h^0\big(X; \mathcal{E}nd(E_{\text{split}})|_X)\big) - h^0\big(X; \mathcal{E}nd\, E|_X\big) = h^1\big(X; \mathcal{E}nd(E_{\text{split}})\big) - h^1\big(X; \mathcal{E}nd\, E|_X\big) = c\, m + d\ ,$$

so we obtain two numbers, the slope c and the intercept d of the dimension difference function. (If we had made the same definition on W_1, we would just get $c = 0$ and $d = \Delta_0 = \Delta_1$.)

Example values on W_1 are tabulated in Table 3.1 and on W_2 in Table 3.2, and we summarise the numerical invariants that we can compute on the spaces W_1, W_2 and W_3:

W_1	height, h', h'', w', w'', Δ_0, Δ_1, h^1
W_2	h'', w'', c, d
W_3	h'', w''

Conjecture 3.20. *On the surface Z_k, we have $w(E) + h(E) = \chi(E) = \frac{h^1 - \Delta_0 - j}{2} + \frac{j}{k}$. This was already mentioned at the end of § 2.7.*

j	p	c	$-d$	(w'', h'')
3	0	0	1	$(2,2)$
3	u	2	1	$(1,2)$
3	zu	2	1	$(0,2)$
3	z^2u	2	1	$(1,2)$
3	$z^2u + u$	2	3	$(0,2)$
3	$z^{-2}v$	3	3	$(2,2)$
3	z^2v	3	3	$(2,2)$
3	$z^2v + u$	3	2	$(1,2)$
3	$z^{-1}v$	4	4	$(2,2)$
3	zv	4	4	$(2,2)$
3	$u + zv$	4	3	$(1,2)$
3	$u + z^{-1}v$	4	4	$(1,2)$
3	$zu + z^{-1}v$	4	4	$(0,2)$
3	v	5	5	$(2,2)$
3	$u + v$	5	5	$(1,2)$
3	$zu + v$	5	4	$(0,2)$
3	$zv + v$	5	5	$(2,2)$

Table 3.2: Example data on W_2.

Direct computation lead us to discover a compact expression for the number $h^1(\mathcal{E}nd\,E)$ on the spaces Z_k and W_1, where $\mathcal{E}$ is either the generic or the split bundle of splitting type j (with $j \geq k$ on Z_k).

Definition 3.21. A power series of the form $g(z) = \sum_{j=0}^{\infty} a_j z^j$ is called a *generating function* for the sequence $(a_j)_{j=0}^{\infty}$. Hence, $a_j = \frac{1}{j!}\frac{d^j g}{dz^j}\Big|_{z=0}$.

In Table 3.3 we present the generating functions for the series $a_j^{X,E} := h^1(\mathcal{E}nd\,E)$ on the spaces Z_k and W_1 for the generic and the split bundle of splitting type j. A few series for special values can be listed explicitly:

- For the split instanton bundle E_j, $j = kn$ on Z_k, $h^1(Z_k; \mathcal{E}nd\,E_j) = n(2nk + k - 2)$.
- For a generic instanton bundle G_j, $j = kn$ on Z_k, we have $h^1(Z_k; \mathcal{E}nd\,G_j) = n(nk+2k-2)-1$ for $k \geq 2$ and $h^1(Z_1; \mathcal{E}nd\,G_j) = j^2$.
- On W_1, we have for the split bundle E_j, $h^1(W_1; \mathcal{E}nd\,E_j) = (4j^3 - j)/3$. This equals the number of coefficients in the generalised extension class $\widetilde{p}$ in Remark 3.3.

Space	Split bundle E_j	Generic bundle G_j
$Z_k,\ k = 2n$	$\dfrac{-z(z^{n+1}+z^n+z+1)}{(z-1)^2(z^k-1)}$	$\dfrac{z^{k+2}-z^3-z^2-z}{(z-1)^2(z^k-1)}$
$Z_k,\ k = 2n+1$	$\dfrac{-z(2z^{n+1}+z+1)}{(z-1)^2(z^k-1)}$	
W_1	$\dfrac{z(z+6)+1}{z(z-1)^4}$	$\dfrac{z(z^4-z^2+2z+1)}{(z-1)^4}$

Table 3.3: Generating functions for $h^1(\mathcal{E}nd\,E)$ on various spaces for the split and the generic bundle of splitting type j (data for G_j only valid for $j \geq k$); the value is the j^{th} coefficient in the Taylor series.

3.6 Moduli

If we restrict our attention to the moduli space of bundles that are extensions of line bundles and do not split on the first infinitesimal neighbourhood, we can apply Equation 1.2 to all three of the spaces W_1, W_2 and W_3.

Definition 3.22. We denote by $\mathfrak{M}(W_i;j)|_1$ the subspace of $\mathfrak{M}(W_i;j)$ consisting of bundles which do not split on the first infinitesimal neighbourhood $\ell^{(1)}$. In terms of the polynomial representative p for a bundle $E(j,p) \in \mathfrak{M}(W_i;j)|_1$, this means that $p(z,u,v) \not\equiv 0 \mod (u,v)^2$.

Remark 3.23. If a bundle $E \in \mathfrak{M}(W_i;j)$ is generic in the sense of Definition 2.23, then $E \in \mathfrak{M}(W_i;j)|_1$. To see this, recall that being generic means that $E|_{\ell^{(1)}} \cong E'|_{\ell^{(1)}}$ implies $E \cong E'$ for any bundle E' and that E is not the split bundle. For contradiction, assume that E splits on $\ell^{(1)}$. But then $E|_{\ell^{(1)}} \cong \overline{E}|_{\ell^{(1)}}$, where $\overline{E} = \mathcal{O}(-j) \oplus \mathcal{O}(j)$ is the split bundle of splitting type j. This would imply that $E \cong \overline{E}$, contradiction. Note however that not every bundle in $\mathfrak{M}(W_i;j)|_1$ is determined on $\ell^{(1)}$, see Proposition 3.28 for an example.

Proposition 3.24. *The part of $\mathfrak{M}(W_i;j)$, $i = 1,2,3$, which consists of extensions that do not split on the first infinitesimal neighbourhood $\ell^{(1)}$ is a projective space of dimension $\gamma_1 - 1$, i.e.*

$$\dim\bigl(\mathfrak{M}(W_i;j)|_1\bigr) = 4j - 5 \quad \textit{for } j \geq 2\,.$$

Moreover, $\mathfrak{M}(W_1;1)|_1$ is empty, $\mathfrak{M}(W_2;1)|_1$ is a point, and $\mathfrak{M}(W_3;1)|_1$ is one-dimensional.

Proof. Let $\mathbb{C}^{\gamma_1}$ be the space of coefficients p_{10s} and p_{01s} of p. By Proposition 3.5 the only isomorphisms on $\ell^{(1)}$ are scaling, and thus $\mathfrak{M}(W_i;j)|_1$ is obtained by projectivising the open subset of generic coefficients of the affine space $\mathbb{C}^{\gamma_1}$.

Now we just compute γ_1 directly as in Equation (1.2): We have $F := E|_\ell \cong \mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$, so

$$\mathcal{E}nd\, F \cong \mathcal{O}_{\mathbb{P}^1}(-2j) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(2j) \cong (\mathcal{E}nd\, F)^\vee \,.$$

Also,

$$N_{\ell,W_1} = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \qquad N_{\ell,W_2} = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1} \qquad N_{\ell,W_3} = \mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \quad ,$$

and by Serre duality,

$$\begin{aligned} \gamma_1 &= h^0\big(\ell; (\mathcal{E}nd\, F \otimes N^*_{\ell,W_i})^\vee \otimes \omega_{\mathbb{P}^1}\big) \\ &= h^0\big(\ell; (\mathcal{E}nd\, F)^\vee \otimes N_{\ell,W_i} \otimes \omega_{\mathbb{P}^1}\big) \\ &= 4(j-1) \quad \text{for all } W_i \text{ and } j \geq 2. \end{aligned}$$

The results for $j = 1$ follow from the same computation. □

Remark 3.25. The result is of Proposition 3.24 is sharp among the spaces W_i in the sense that it is only true for $i = 1, 2, 3$; for any $i > 3$ the value of γ_1 is greater than $4(j-1)$.

3.6.1 Bundles on W_1

The space $W_1 := \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\big)$, also known the "simple flop", was already studied in passing in [BGK2]. Here we present a more detailed treatment, which also illustrates the connection between the parameter space of bundles and the moduli of isomorphism classes of extensions. We give an explicit description of $\mathfrak{M}(W_1; j)$ for small j:

Proposition 3.26 (Moduli on W_1 for $j = 0, 1$).

Let E be a bundle on W_1 of splitting type $j = 0$. Then $E \cong \mathcal{O}^{\oplus 2}$, i.e. E is trivial.

Let E be a bundle on W_1 of splitting type $j = 1$. Then $E \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$, i.e. E splits.

Proof. This follows immediately from Proposition 3.1: On W_1 for $j = 0$ or $j = 1$ we can always write the extension as $p = 0$, so every rank-2 bundle of this splitting type is the split bundle. □

Remark 3.27. Thus there are no "generic" bundles of splitting type 0 or 1 on W_1, i.e. no bundles that do not split on $\ell^{(1)}$.

Proposition 3.28 (Moduli on W_1 for $j = 2$). *Let E be a rank-2 vector bundle on W_1 given by the transition matrix*

$$\begin{pmatrix} z^2 & p(z,u,v) \\ 0 & z^{-2} \end{pmatrix} .$$

The space of isomorphism classes of such bundles is the quotient of the space

$$\left\{ \left(p_{010}, p_{011}, p_{100}, p_{101}\right), p_{021}, p_{201}, p_{111} \right\} \subseteq \mathbb{C}^7$$

with the Euclidean topology by certain relations presented at the end of the proof.

Proof. By Proposition 3.1, a bundle on W_1 with splitting type $j = 2$ is determined by its extension class

$$p(z,u,v) = (p_{010} + p_{011}z)u + (p_{100} + p_{101}z)v + p_{021}zu^2 + p_{201}zv^2 + p_{111}zuv .$$

Suppose we have two such bundles E and E', given respectively by extension classes p and q. We write $p|_{\ell^{(1)}}$ for the restriction to the first formal neighbourhood, i.e. to terms that have total degree ≤ 1 in u, v; so

$$p|_{\ell^{(1)}}(z,u,v) = (p_{010} + p_{011}z)u + (p_{100} + p_{101}z)v ,$$

and similarly for q.

If $E \cong E'$, then by Proposition 3.5, $p|_{\ell^{(1)}} = \lambda q|_{\ell^{(1)}}$ for some $\lambda \in \mathbb{C}^\times$, and by rescaling we may assume that $\lambda = 1$. (We return to this point at the end of the proof.) Now we identify sufficient conditions for an isomorphism: The isomorphism, if it exists, can be written as

$$\begin{aligned} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} z^2 & p \\ 0 & z^{-2} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} z^{-2} & -q \\ 0 & z^2 \end{pmatrix} \\ &= \begin{pmatrix} A + z^{-2}pC & z^4B + z^2(pD - qA) - pqC \\ z^{-4}C & D - z^{-2}qC \end{pmatrix} , \end{aligned}$$

where A, B, C, D are holomorphic on $U \cap \widehat{\ell}$ and $\alpha, \beta, \gamma, \delta$ on $V \cap \widehat{\ell}$, i.e. power series in (z, u, v) or (z^{-1}, zu, zv), respectively. As usual we write $A = \sum a_{trs} z^s u^r v^t$ etc.

First, the condition that the $(2,1)$-entry be holomorphic in (z^{-1}, zu, zv) implies

$$C = \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{r+t+4} c_{trs} z^s u^r v^t .$$

Next, the condition of holomorphy of the diagonal entries yields the following relations for the terms of the power series A and D:

$$-\begin{pmatrix} a_{012} \\ a_{013} \\ a_{014} \\ a_{015} \end{pmatrix} = \begin{pmatrix} p_{011} & p_{010} & 0 & 0 \\ 0 & p_{011} & p_{010} & 0 \\ 0 & 0 & p_{011} & p_{010} \\ 0 & 0 & 0 & p_{011} \end{pmatrix} \begin{pmatrix} c_{001} \\ c_{002} \\ c_{003} \\ c_{004} \end{pmatrix}$$

$$-\begin{pmatrix} a_{102} \\ a_{103} \\ a_{104} \\ a_{105} \end{pmatrix} = \begin{pmatrix} p_{101} & p_{100} & 0 & 0 \\ 0 & p_{101} & p_{100} & 0 \\ 0 & 0 & p_{101} & p_{100} \\ 0 & 0 & 0 & p_{101} \end{pmatrix} \begin{pmatrix} c_{001} \\ c_{002} \\ c_{003} \\ c_{004} \end{pmatrix}$$

$$-\begin{pmatrix} a_{023} \\ a_{024} \\ a_{025} \\ a_{026} \end{pmatrix} = \begin{pmatrix} p_{011} & p_{010} & 0 & 0 \\ 0 & p_{011} & p_{010} & 0 \\ 0 & 0 & p_{011} & p_{010} \\ 0 & 0 & 0 & p_{011} \end{pmatrix} \begin{pmatrix} c_{012} \\ c_{013} \\ c_{014} \\ c_{015} \end{pmatrix} + \begin{pmatrix} p_{021} & 0 & 0 & 0 \\ 0 & p_{021} & 0 & 0 \\ 0 & 0 & p_{021} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{002} \\ c_{003} \\ c_{004} \\ c_{005} \end{pmatrix}$$

$$-\begin{pmatrix} a_{203} \\ a_{204} \\ a_{205} \\ a_{206} \end{pmatrix} = \begin{pmatrix} p_{101} & p_{100} & 0 & 0 \\ 0 & p_{101} & p_{100} & 0 \\ 0 & 0 & p_{101} & p_{100} \\ 0 & 0 & 0 & p_{101} \end{pmatrix} \begin{pmatrix} c_{102} \\ c_{103} \\ c_{104} \\ c_{105} \end{pmatrix} + \begin{pmatrix} p_{201} & 0 & 0 & 0 \\ 0 & p_{201} & 0 & 0 \\ 0 & 0 & p_{201} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{002} \\ c_{003} \\ c_{004} \\ c_{005} \end{pmatrix}$$

$$-\begin{pmatrix} a_{113} \\ a_{114} \\ a_{115} \\ a_{116} \end{pmatrix} = \begin{pmatrix} p_{101} & p_{100} & 0 & 0 \\ 0 & p_{101} & p_{100} & 0 \\ 0 & 0 & p_{101} & p_{100} \\ 0 & 0 & 0 & p_{101} \end{pmatrix} \begin{pmatrix} c_{012} \\ c_{013} \\ c_{014} \\ c_{015} \end{pmatrix} + \begin{pmatrix} p_{011} & p_{010} & 0 & 0 \\ 0 & p_{011} & p_{010} & 0 \\ 0 & 0 & p_{011} & p_{010} \\ 0 & 0 & 0 & p_{011} \end{pmatrix} \begin{pmatrix} c_{102} \\ c_{103} \\ c_{104} \\ c_{105} \end{pmatrix} + \begin{pmatrix} p_{111} & 0 & 0 & 0 \\ 0 & p_{111} & 0 & 0 \\ 0 & 0 & p_{111} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{002} \\ c_{003} \\ c_{004} \\ c_{005} \end{pmatrix}$$

$$\begin{pmatrix} d_{012} \\ d_{013} \\ d_{014} \\ d_{015} \end{pmatrix} = \begin{pmatrix} p_{011} & p_{010} & 0 & 0 \\ 0 & p_{011} & p_{010} & 0 \\ 0 & 0 & p_{011} & p_{010} \\ 0 & 0 & 0 & p_{011} \end{pmatrix} \begin{pmatrix} c_{001} \\ c_{002} \\ c_{003} \\ c_{004} \end{pmatrix}$$

$$\begin{pmatrix} d_{102} \\ d_{103} \\ d_{104} \\ d_{105} \end{pmatrix} = \begin{pmatrix} p_{101} & p_{100} & 0 & 0 \\ 0 & p_{101} & p_{100} & 0 \\ 0 & 0 & p_{101} & p_{100} \\ 0 & 0 & 0 & p_{101} \end{pmatrix} \begin{pmatrix} c_{001} \\ c_{002} \\ c_{003} \\ c_{004} \end{pmatrix}$$

$$\begin{pmatrix} d_{023} \\ d_{024} \\ d_{025} \\ d_{026} \end{pmatrix} = \begin{pmatrix} p_{011} & p_{010} & 0 & 0 \\ 0 & p_{011} & p_{010} & 0 \\ 0 & 0 & p_{011} & p_{010} \\ 0 & 0 & 0 & p_{011} \end{pmatrix} \begin{pmatrix} c_{012} \\ c_{013} \\ c_{014} \\ c_{015} \end{pmatrix} + \begin{pmatrix} p_{021} & 0 & 0 & 0 \\ 0 & p_{021} & 0 & 0 \\ 0 & 0 & p_{021} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{002} \\ c_{003} \\ c_{004} \\ c_{005} \end{pmatrix}$$

$$\begin{pmatrix} d_{203} \\ d_{204} \\ d_{205} \\ d_{206} \end{pmatrix} = \begin{pmatrix} p_{101} & p_{100} & 0 & 0 \\ 0 & p_{101} & p_{100} & 0 \\ 0 & 0 & p_{101} & p_{100} \\ 0 & 0 & 0 & p_{101} \end{pmatrix} \begin{pmatrix} c_{102} \\ c_{103} \\ c_{104} \\ c_{105} \end{pmatrix} + \begin{pmatrix} p_{201} & 0 & 0 & 0 \\ 0 & p_{201} & 0 & 0 \\ 0 & 0 & p_{201} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{002} \\ c_{003} \\ c_{004} \\ c_{005} \end{pmatrix}$$

$$\begin{pmatrix} d_{113} \\ d_{114} \\ d_{115} \\ d_{116} \end{pmatrix} = \begin{pmatrix} p_{101} & p_{100} & 0 & 0 \\ 0 & p_{101} & p_{100} & 0 \\ 0 & 0 & p_{101} & p_{100} \\ 0 & 0 & 0 & p_{101} \end{pmatrix} \begin{pmatrix} c_{012} \\ c_{013} \\ c_{014} \\ c_{015} \end{pmatrix} + \begin{pmatrix} p_{011} & p_{010} & 0 & 0 \\ 0 & p_{011} & p_{010} & 0 \\ 0 & 0 & p_{011} & p_{010} \\ 0 & 0 & 0 & p_{011} \end{pmatrix} \begin{pmatrix} c_{102} \\ c_{103} \\ c_{104} \\ c_{105} \end{pmatrix} + \begin{pmatrix} p_{111} & 0 & 0 & 0 \\ 0 & p_{111} & 0 & 0 \\ 0 & 0 & p_{111} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{002} \\ c_{003} \\ c_{004} \\ c_{005} \end{pmatrix}$$

Lastly, the $(1,2)$-entry,

$$z^4B + z^2(pD - qA) - pqC\,,$$

has to be made holomorphic in (z^{-1}, zu, zv). Since z^4B can be chosen to cancel higher terms, we only need to consider those terms $z^s u^r v^t$ with $s > r + t$ and $s < 4$ in the expression $z^2(pD - qA) - pqC$.

Recall that A and D are of the form $1+u(\ldots)+v(\ldots)$ and that we have already set $p|_{\ell^{(1)}} = q|_{\ell^{(1)}}$. Thus the coefficients of u and v are zero. It remains to find the coefficients of the terms u^2, v^2 and uv. From this we get three equations:

$$\begin{aligned}
0 &= (p_{021} - q_{021}) - c_{003}p_{010}^2 - 2c_{002}p_{010}p_{011} - c_{001}p_{011}^2 \\
&\quad + p_{010}(d_{011} - a_{011}) + p_{011}(d_{010} - a_{010}) &(3.14)\\
0 &= (p_{201} - q_{201}) - c_{003}p_{100}^2 - 2c_{002}p_{100}p_{101} - c_{001}p_{101}^2 \\
&\quad + p_{100}(d_{101} - a_{101}) + p_{101}(d_{100} - a_{100}) &(3.15)\\
0 &= (p_{111} - q_{111}) - 2c_{003}p_{010}p_{101} - 2c_{002}(p_{010}p_{101} + p_{011}p_{100}) \\
&\quad - 2c_{001}p_{011}p_{101} + p_{100}(d_{011} - a_{011}) + p_{010}(d_{101} - a_{101}) \\
&\quad + p_{101}(d_{010} - a_{010}) + p_{011}(d_{100} - a_{100}) &(3.16)
\end{aligned}$$

Finally we describe the moduli of extensions of splitting type $j = 2$ as the space of coefficients

$$X := \Big\{ \big(p_{010}, p_{011}, p_{100}, p_{101}\big), p_{021}, p_{201}, p_{111} \Big\}$$

modulo relations that we infer from the above equations:

1. If $(p_{010}, p_{011}) \neq (0,0)$, Equations (3.14) and (3.16) can be solved for any $p_{021}, p_{201}, p_{111}$.
2. If $(p_{100}, p_{101}) \neq (0,0)$, Equations (3.15) and (3.16) can be solved for any $p_{021}, p_{201}, p_{111}$.
3. If all first-order coefficients vanish, the three equations imply that all the second-order coefficients are equal.

Thus the moduli space is a quotient space of the Euclidean $\mathbb{C}^7$ by a number of relations, which we describe in four parts:

1. The generic set $S := \{p_{010}, p_{011}, p_{100}, p_{101}\}/\mathbb{C}^\times$, where $(p_{010}, p_{011}) \neq (0,0)$ and $(p_{100}, p_{101}) \neq (0,0)$,

2. the set $T_1 := \left\{(0,0,p_{100},p_{101}),p_{021}\right\}/\mathbb{C}^\times$, where $p_{021} \in \mathbb{C}$ and $(p_{100},p_{101}) \neq (0,0)$,
3. the set $T_2 := \left\{(p_{010},p_{011},0,0),p_{201}\right\}/\mathbb{C}^\times$, where $p_{201} \in \mathbb{C}$ and $(p_{010},p_{011}) \neq (0,0)$,
4. the set $P := \left\{(0,0,0,0),p_{021},p_{201},p_{111}\right\}/\mathbb{C}^\times \cong \mathbb{CP}^2$, where $(p_{021},p_{201},p_{111}) \neq (0,0,0)$, and
5. the split bundle (all $p_{trs} = 0$).

$\square$

Conclusion: The moduli of extensions of splitting type $j = 2$ on W_1 is

$$\mathfrak{M}(W_1;2) = S \sqcup T_1 \sqcup T_2 \sqcup P \sqcup \{*\} \,,$$

where the generic set in the sense of Definition 2.23 is

$$S = \left\{[p_{010} : p_{011} : p_{100} : p_{101}] \in \mathbb{C}P^3 : (p_{010},p_{011}) \neq (0,0) \text{ and } (p_{100},p_{101}) \neq (0,0)\right\} .$$

Furthermore,

$$\begin{aligned} T_1 &= \left\{[p_{100} : p_{101} : p_{021}] \smallsetminus [0:0:1]\right\} \cong \mathbb{C}P^2 \smallsetminus \{*\} \text{ and} \\ T_2 &= \left\{[p_{010} : p_{011} : p_{201}] \smallsetminus [0:0:1]\right\} \cong \mathbb{C}P^2 \smallsetminus \{*\} . \end{aligned}$$

The generic set S has dimension $4 \cdot 2 - 5 = 3$, the same as $\dim \mathfrak{W}_1;2|_1$ in Proposition 3.24, and the set S is a *proper* subset of $\mathfrak{W}_1;2|_1 \cong \mathbb{CP}^3$ (see Remark 3.23), since the bundles with $(p_{100},p_{101}) = (0,0)$ or $(p_{010},p_{011}) = (0,0)$ are *not* generic, and those form a closed subset. Note that the generalised extension class for $j = 2$ is

$$\begin{aligned} \widetilde{p}(z,u,v) = p_{0,0,-1}z^{-1} + p_{000} + p_{001}z + (p_{010} + p_{011}z)u \\ + (p_{100} + p_{101}z)v + p_{021}zu^2 + p_{201}zv^2 + p_{111}zuv \,, \end{aligned}$$

which has $\gamma = 10$ coefficients, in accord with Remark 3.3, the first three of which give deformations along ℓ into bundles of lower splitting type.

$p(z,u,v)$	cpt. of $\mathfrak{M}(W_1;2)$	h	h'	w'	h''	w''	h^1	Δ_1	Δ_0
0	$\{*\}$	1	1	3	1	3	10	0	0
zuv	P	1	1	3	1	3	9	1	1
zu^2	P	1	1	2	1	3	9	1	1
v	T_1	1	1	3	1	1	7	3	3
u	T_2	1	1	1	1	3	7	3	3
$u+v$	S	1	1	1	1	1	7	3	3
$u+zv$	S	1	1	1	1	1	6	4	4

Table 3.4: Numerical invariants of several bundles on W_1 of splitting type $j=2$.

We can express this result compactly as follows: The moduli $\mathfrak{M}(W_1;2)$ is the space of orbits in $\mathbb{C}^7$ of the action

$$\begin{pmatrix} p_{010} \\ p_{011} \\ p_{100} \\ p_{101} \\ p_{021} \\ p_{111} \\ p_{201} \end{pmatrix} \mapsto \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ \alpha_1 & \alpha_2 & 0 & 0 & \lambda & 0 & 0 \\ \alpha_3 & \alpha_4 & \beta_3 & \beta_4 & 0 & \lambda & 0 \\ 0 & 0 & \beta_1 & \beta_2 & 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} p_{010} \\ p_{011} \\ p_{100} \\ p_{101} \\ p_{021} \\ p_{111} \\ p_{201} \end{pmatrix},$$

where $\lambda \in \mathbb{C}^\times$ and $\alpha_i, \beta_i \in \mathbb{C}$ for $i = 1, \ldots, 4$. Note that the group G that acts is not reductive, and thus the quotient is not amenable to standard GIT techniques (although Kirwan has been working on quotients by non-reductive groups, cf. for instance [Kir09], and it would be interesting to explore our situation in the context of that work). Explicitly, G is given as the extension $0 \to (\mathbb{C}^\times, \times) \to G \to (\mathbb{C}, +)^8 \to 0$.

We see directly that the restriction of the action of G to the subspace $\mathbb{C}^4$ spanned by the first four coefficients $\{p_{010}, p_{011}, p_{100}, p_{101}\}$ reduces to $\mathbb{C}^\times$, which acts faithfully, and this subspace is the largest subset whose quotient by the G-action is Hausdorff. The set S of generic bundles that we identified above is a Zariski-open subset of this quotient.

The numerical invariants h, h', w', h'', w'' and h^1 help distinguish the different types of bundles, and they are tabulated in Table 3.4. The table gives also the additional numbers Δ_0 and Δ_1, but recall that they are determined by h^1. While those numerical invariants are not quite sufficient to give $\mathfrak{M}(W_1;2)$ a Hausdorff decomposition, it does suffice to identify the generic set S, which is the one where the sum of the h', w', h'', w'' is minimal.

Table 3.4 exhibits another phenomenon: The two bundles given by zu^2 and zuv are clearly in the same part of the moduli and related by a change of coordinates, yet the partial invariants w' and w'' differ; the "correct" value is given by $w(E_{\{u+v=0\}}) = 3$. To make the use of partial invariants general, we could devise a family version parametrised by $[\lambda : \mu] \in \mathbb{P}^1$ computing $w(E_{\{\lambda u+\mu v=0\}})$. The number h^1 actually provides a finer invariant than needed, as the generic set consists of those bundles with $h^1 \leq 7$.

3.6.2 Bundles on W_2

The crucial difference between W_2 and W_1 is that $N^*_{\ell,W_2} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}$ is not ample. We see both from the form of the canonical extension class in Proposition 3.1 and from the dimension count (1.2) that the parameter space for extensions $p(z,u,v) = \sum p_{trs} z^s u^r v^t$ is infinite-dimensional, and it is clear that there exist non-algebraic bundles, e.g. on the subspace $\mathrm{Tot}\big(0 \oplus \mathcal{O}_{\mathbb{P}^1}\big)$. Nonetheless, we saw in Theorem 3.11 that every bundle on W_2 is still filtrable.

This means that the moduli of all rank-2 bundles with vanishing first Chern class is still a union of moduli $\mathfrak{M}(W_2; j)$ of extensions of fixed splitting type j. Even though each $\mathfrak{M}(W_2; j)$ is now in some sense infinite, we can still attempt to describe it. We start with a few moduli $\mathfrak{M}(W_2; j)$ for small j. The case $j = 0$ is easy:

Proposition 3.29 (Moduli on W_2 for $j = 0$)**.** *Let E be a bundle on W_2 of splitting type $j = 0$. Then $E \cong \mathcal{O}^{\oplus 2}$, i.e. E is trivial.*

Proof. This follows immediately from Proposition 3.1: On W_2 for $j = 0$ we can always write the extension as $p = 0$, so every rank-2 bundle of this splitting type is trivial. □

For $j = 1$, Proposition 3.24 shows that there is only one single generic bundle. The full space $\mathfrak{M}(W_2; 1)$ can be described as follows. Substituting $j = 1$ into Proposition 3.1, we see that the polynomial p must be of the form $p(z,u,v) = \sum_{t=1}^{\infty} p_{t00} v^t$.

Proposition 3.30 (Moduli on W_2 for $j = 1$)**.** *Let E_p, E_q be two bundles on W_2 of splitting type $j = 1$ determined by polynomials p, q, respectively. E_p and E_q are isomorphic if and only if one of the following to conditions hold:*

- *If $p \equiv 0 \equiv 0$; or*
- *if $p_1 \neq 0 \neq q_1$; or*
- *if $p_i = 0 = q_i$ for $i \geq 1$ and $p_{i+1} = q_{i+1} \neq 0$.*

Proof. We determine conditions for E_q and E_q to be isomorphic by rerunning the proof of Proposition 3.28: We need transformation matrices

$$\begin{aligned}\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} &= \begin{pmatrix}z & p \\ 0 & z^{-1}\end{pmatrix}\begin{pmatrix}A & B \\ C & D\end{pmatrix}\begin{pmatrix}z^{-1} & -q \\ 0 & z\end{pmatrix} \\ &= \begin{pmatrix}A + z^{-1}pC & z^2B + z(pD - qA) - pqC \\ z^{-2}C & D - z^{-1}qC\end{pmatrix},\end{aligned}$$

where A, B, C, D are holomorphic on $U \cap \hat{\ell}$ and $\alpha, \beta, \gamma, \delta$ on $V \cap \hat{\ell}$, i.e. formal power series in (z, u, v) or (z^{-1}, z^2u, v), respectively.

First, the condition that the $(2,1)$-entry be holomorphic in (z^{-1}, z^2u, v) implies

$$C = \sum_{t=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{2r+2} c_{trs}z^s u^r v^t \ .$$

Second, we obtain another set of equations like those on page 75, which we omit here to avoid cluttering the presentation.

Finally, the $(1,2)$-entry

$$z^2B + z(pD - qA) - pqC$$

has to be made holomorphic in (z^{-1}, z^2u, v). Since z^2B can be chosen to cancel higher terms, we only need to consider those terms $z^s u^r v^t$ with $s > 2r$ and $s < 2$ in the expression $z(pD - qA) - pqC$. This leaves only the terms zv^t with $t \geq 1$. Moreover, by Proposition 3.5, $E \cong E'$ implies that we can scale q such that $p|_{\ell^{(1)}} = q|_{\ell^{(1)}}$. Thus in fact we only need to consider $t \geq 2$.

$$\begin{aligned}&z\Big(\big(p_1v + p_2v^2 + p_3v^3 + \cdots\big)\big(1 + u(\ldots) + v(d_{100} + d_{101}z + d_{102}z^2 + \cdots)\big) \\ &\quad -\big(p_1v + q_2v^2 + q_3v^3 + \cdots\big)\big(1 + u(\ldots) + v(a_{100} + a_{101}z + a_{102}z^2 + \cdots)\big)\Big) \\ &\quad -\big(p_1v + p_2v^2 + p_3v^3 + \cdots\big)\big(p_1v + q_2v^2 + q_3v^3 + \cdots\big)\sum_{t=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{2r+2} c_{trs}z^s u^r v^t\end{aligned}$$

We obtain an infinite series of equations. The first few are:

$$\begin{aligned}&\text{For } v^2z: && p_1(d_{100} - a_{100}) + (p_2 - q_2) - p_1^2c_{001} = 0\ . \\ &\text{For } v^3z: && p_1(d_{200} - a_{200}) + (p_2d_{100} - q_2a_{100}) + (p_3 - q_3) - p_1^2c_{101} - (p_1q_2 + p_2q_1)c_{001} = 0\ . \\ &\quad\vdots && \quad\vdots\end{aligned}$$

From this we obtain the following infinite list of families of extensions:

- The split bundle, $p = 0$.
- One generic bundle $p = v$, isomorphic to all $p = p_1 v + \sum_{t\geq 2} p_2 v^2$ for $p_1 \neq 0$.
- A family $p = p_2 v^2$ with $p_2 \in \mathbb{C}^\times$, each member being isomorphic to $p = p_2 v^2 + \sum_{t\geq 3} p_t v^t$.
- A family $p = p_3 v^3$ with $p_3 \in \mathbb{C}^\times$, each member being isomorphic to $p = p_3 v^3 + \sum_{t\geq 4} p_t v^t$.
- …

□

3.6.3 Bundles on W_3

On $W_3 := \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)\big)$ the conormal sheaf N^*_{ℓ,W_3} is not ample, but unlike on W_2 it is not even possible to express every vector bundle as a filtration. In particular, there are rank-2 bundles that are not extensions of line bundles, and the transition functions need not be algebraic (e.g. on the subset $\mathrm{Tot}\big(0 \oplus \mathcal{O}_{\mathbb{P}^1}(1)\big)$).

Note however that for purely dimensional reasons,

$$h^2\big(\ell;\ \mathcal{E}nd\, F \otimes S^n(N^*_{\ell,W})\big) = 0$$

for every vector bundle (in fact, coherent sheaf) F on ℓ, and that it is thus possible to extend F to a bundle E on $\widehat{\ell}$ such that $E|_\ell = F$; however, just like on W_2 there are now infinitely many non-zero terms in the sum in Equation (1.2), i.e. infinitely many directions in which to extend.

If we only consider rank-2 bundles that are extensions of the form (2.1), we still know from Proposition 3.5 that the space of extensions modulo isomorphisms has a generic set of dimension $4j - 5$.

When $j = 0$ we have $p|_{\ell(1)} \equiv 0$ by Proposition 3.1. However, there are many non-equivalent bundles on W_3 even for $j = 0$. For $j = 1$, we have a one-dimensional family of generic extensions given by $p_{100} v + p_{1,0,-1} z^{-1} v$ (modulo projectivisation).

3.6.4 Structure on the moduli

Let us track back a little. Initially, we had no reason to expect an inherent structure *a priori* on the moduli of rank-2 bundles. We divided the problem by decomposing $\mathfrak{M}(X) = \bigsqcup_{j\geq 0} \mathfrak{M}(X; j)$ as sets, where X is one of Z_k, W_1 or W_2. We then equipped each piece $\mathfrak{M}(X; j)$ with the topology of the quotient of a vector space (Definition 2.21).

We can take another view on this construction. The abstract vector space $\operatorname{Ext}^1_X\big(\mathcal{O}(j), \mathcal{O}(-j)\big)$, for $X = W_1, W_2$, say, can be realised as the space of coefficients $\{\text{coefficients } p_{trs}\} \cong \mathbb{C}^{\gamma_+}$. Here γ_+ is the number of coefficients in p according to Proposition 3.1, and it is precisely the dimension count explained in Remark 1.2 in the introduction:

$$\gamma_+ = \sum_{n=1}^{\infty} h^1\big(\ell;\ \mathcal{E}nd\, F \otimes S^n(N^*_{\ell,W_i})\big) \leq \infty\ .$$

The quotient topology is still not Hausdorff, since for instance the split bundle $\mathcal{O}(-j) \oplus \mathcal{O}(j)$ is "near" every bundle in this topology (i.e. the split bundle deforms into every other bundle by taking $E(j, \epsilon p)$ and letting ϵ go to zero). However, we showed that the subset of bundles that do not split on the first neighbourhood, and which we denoted by $\mathfrak{M}(W_i; j)|_1$, form a projective space of dimension $4j - 5$.

We showed in § 2.6.2 that on the surfaces Z_k the numerical invariants (h_k, w_k) stratify the spaces $\mathfrak{M}(Z_k; j)$ into Hausdorff components. While we do not have the analogous result for W_1 and W_2, we remarked that on W_1, the width always vanishes. However, our concrete example of $\mathfrak{M}(W_1; 3)$ suggests that there does indeed exist a full stratification into components that are not only Hausdorff but also possess the structure of a manifold or a variety, and we proposed several numerical invariants which can tell the strata apart. The key point is that we parametrise bundles by a vector space of coefficients of the polynomial p, and that the question whether two bundles are isomorphic or what their numerical invariants are only depends on whether a coefficient is zero or non-zero, and thus we believe that there should be a stratification of $\mathfrak{M}(W_i; j)$, $i = 1, 2$, in which all strata should have the structure of a projective space with linear subspaces removed.

Generalisations. After having studied numerous examples in detail, we can make a few generalising remarks. If $\ell \cong \mathbb{P}^1$ is a line inside any complex space W and $N^*_{\ell,W}$ is ample, then as discussed in § 2.3, bundles on an analytic neighbourhood $N(\ell)$ are determined on a finite infinitesimal neighbourhood $\ell^{(M)}$, and the situation is modelled on

$$W_{\mathbf{k}} := \operatorname{Tot}\big(N_{\ell,W}\big) = \operatorname{Tot}\Big(\bigoplus_i \mathcal{O}_{\mathbb{P}^1}(-k_i)\Big)\ ,$$

where $k_i > 0$ for all i. Bundles on $W_{\mathbf{k}}$ are filtered and algebraic by [BGK2, Theorem 3.2]. A rank-2 bundle E on $W_{\mathbf{k}}$ still splits as $\mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$ on ℓ, and the dimension of the generic set of the moduli space $\mathfrak{M}(j)$ of extensions of splitting type j modulo bundle isomorphisms can be calculated as the γ_1-term in Equation (1.2). Each surface $D_i := \operatorname{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-k_i)\big)$ in $W_{\mathbf{k}}$ is now of the form Z_{k_i} as studied in [BGK1], and a bundle E is generic if it is generic on each D_i. Expressed conversely, if a bundle E given by the extension class p, which only has terms of order 1 in the fibre directions, is *not* generic, then its restriction to some D_i will be the split bundle, which can be identified by its numerical invariants according to [BGK1]. Among all bundles which have only terms of first order in the fibre directions in their extension class, the generic ones are precisely those for which the sum of all partial invariants $\sum_i \big(h(E|_{D_i}) + w(E|_{D_i}) \big)$ is minimal.

Theorem 3.31. *There exists a two-parameter family of embeddings $\Phi_{s,t}\colon \mathfrak{M}(j) \hookrightarrow \mathfrak{M}(j+1)$, $(s,t) \in \mathbb{P}^1 \times \mathbb{P}^1$. Bundles in the image are of splitting type $j+1$ and split on the second infinitesimal neighbourhood.*

Proof. Suppose E is a bundle on W_1 of splitting type j given by the polynomial $p(z,u,v)$. For $[a_1 : b_1], [a_2, b_2] \in (\mathbb{P}^1)^2$, there is a map

$$\Phi_{[a_1:b_1],[a_2,b_2]}\colon \mathfrak{M}(j) \to \mathfrak{M}(j+1)$$

which is the composite of two elementary transformations over the divisors $D_i = \{a_i u + b_i v\}$, $i = 1, 2$ followed by a twist by $\mathcal{O}(-1)$:

$$\Phi_{[a_1:b_1],[a_2,b_2]}(E) = (\mathrm{Elm}_{D_2} \circ \mathrm{Elm}_{D_1})(E) \otimes \mathcal{O}(-1)$$

If the bundle E is of splitting type j and given by the polynomial p, then $E' := \Phi_{[a_1:b_1],[a_2,b_2]}(E)$ is given by $z(a_1 u + b_1 v)(a_2 u + b_2 v)p$. Furthermore, $E'|_\ell \cong \mathcal{O}_{\mathbb{P}^1}(-j-1) \oplus \mathcal{O}_{\mathbb{P}^1}(j+1)$, so E' is of splitting type $j+1$, and E' fits into the exact sequence

$$0 \longrightarrow \mathcal{O}(-j-1) \longrightarrow E' \longrightarrow \mathcal{O}(j+1) \longrightarrow 0\,.$$

By construction, $E'|_{\ell^{(2)}} \cong \mathcal{O}_{\ell^{(2)}}(-j-1) \oplus \mathcal{O}_{\ell^{(2)}}(j+1)$, that is, bundles in the image split on the second infinitesimal neighbourhood of ℓ. □

Bibliography

[BF97] Kai Behrend and Barbara Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), no. 1, 45–88.

[BGK1] Eduardo Ballico, Elizabeth Gasparim, and Thomas Köppe, *Vector bundles near negative curves: moduli and local Euler characteristic*, Comm. Algebra **37** (2009), no. 8, 2688–2713.

[BGK2] Eduardo Ballico, Elizabeth Gasparim, and Thomas Köppe, *Local moduli of holomorphic bundles*, J. Pure Appl. Algebra **213** (2009), 397–408.

[Bl96] Raimund Blache, *Chern classes and Hirzebruch-Riemann-Roch theorem for coherent sheaves on complex-projective orbifolds with isolated singularities*, Math. Z. **222** (1996), no. 1, 7–57.

[dJP00] Theo de Jong and Gerhard Pfister, *Local analytic geometry: Basic theory and applications*, Vieweg Verlag, 2000.

[Ei95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry.

[FK74] O. Forster and K. Knorr, *Über die Deformationen von Vektorraumbündeln auf kompakten komplexen Räumen*, Math. Ann. **209** (1974), 291–346.

[Ga97] Elizabeth Gasparim, *Holomorphic bundles on $\mathcal{O}(-k)$ are algebraic*, Comm. Algebra **25** (1997), no. 9, 3001–3009.

[Ga08] Elizabeth Gasparim, *Atiyah-Jones conjecture for rational surfaces*, Advances Math. **218** (2008), 1027–1050.

[GK] Elizabeth Gasparim and Thomas Köppe, *Sheaves on singular varieties*, Proc. Singularities in Aahrus, in honour of Andrew du Plessis, to appear.

[GKM] Elizabeth Gasparim, Thomas Köppe, and Pushan Majumdar, *Local holomorphic Euler characteristic and instanton decay*, Pure Appl. Math. Q. **4** (2008), no. 2, 161–179, Special Issue: In honor of Fedya Bogomolov, Part 1.

[GPR94] H. Grauert, Th. Peternell, and R. Remmert (eds.), *Several complex variables VII*, Springer-Verlag, 1994.

[Gr61] Alexander Grothendieck, *Élements de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I.*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 5–167.

[GS05] Elizabeth Gasparim and Irena Swanson, *Computing instanton numbers of curve singularities*, J. Symbolic Computation **40** (2005), no. 2, 965–978.

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, no. 52, Springer, 1977.

[Har80] Robin Hartshorne, *Stable reflexive sheaves*, Math. Ann. **254** (1980), no. 2, 121–176.

[HL97] Daniel Huybrechts and Manfred Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[Jim92] Jesús Jiménez, *Contraction of nonsingular curves*, Duke Math. J. **65** (1992), no. 2, 313–332.

[Kir09] Frances Kirwan, *Quotients by non-reductive algebraic group actions*, Moduli spaces and vector bundles, London Math. Soc. Lecture Note Ser., vol. 359, Cambridge Univ. Press, Cambridge, 2009, pp. 311–366.

[Lan02] Serge Lang, *Algebra*, third ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002.

[M2] Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/.

[Pet81] Thomas Peternell, *Vektorraumbündel in der Nähe von kompakten komplexen Unterräumen*, Math. Ann. **257** (1981), no. 1, 111–134.

[Pet82] Thomas Peternell, *Vektorraumbündel in der Nähe von exzeptionellen Unterräumen – das Modulproblem*, J. Reine Angew. Math. **336** (1982), 110–123.